

1. a) Figure 1.1 illustrates an RLC circuit. The capacitor has capacitance C , the inductor has inductance L and the resistor resistance R . The input is the applied voltage $v_i(t)$ and the output is the voltage across the capacitor and resistor $v_o(t)$.

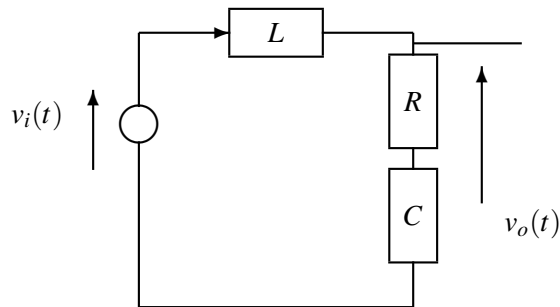


Figure 1.1

- i) Determine $G(s)$, the transfer function relating v_o to v_i . [4]
- ii) Let $v_i(t)$ be a unit step applied at $t = 0$. Use the final value theorem, which should be stated, to find the steady-state value of $v_o(t)$. [5]
- iii) Derive the value of R so that $G(s)$ is marginally stable. What is the frequency of oscillations? Give your answer in terms of L and C . [5]
- b) In Figure 1.2 below, $G(s) = \frac{s+1}{s-1}$ and K is a variable gain.
- i) Sketch the locus of the closed-loop poles for $0 \leq K < \infty$. [5]
- ii) Using the gain criterion, find the value of K for which the closed-loop is marginally stable. [4]
- iii) Find the range of $K \geq 0$ for which the closed-loop is stable. [4]
- c) In Figure 1.2 below, $G(s) = \frac{1}{s-0.5}$ and K is a variable gain.
- i) Draw the Nyquist diagram of $G(s)$ indicating real-axis intercepts. [5]
- ii) Take $K = 0.25$. Use the Nyquist criterion, which should be stated, to determine the number of unstable closed-loop poles. [4]
- iii) Take $K = 1$. Use the Nyquist criterion to show that the closed-loop is stable. Comment on the gain margin. [4]

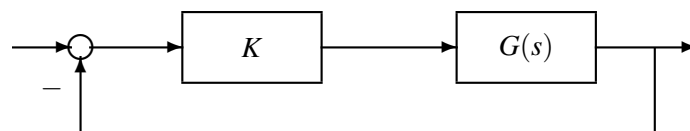


Figure 1.2

2. Let

$$G(s) = \frac{1}{s+1}$$

and consider the feedback loop shown in Figure 2 below.

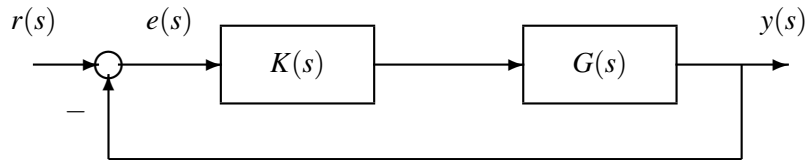


Figure 2

A feedback compensator $K(s)$ is required such that the following design specifications are satisfied:

- (i) The closed-loop is stable.
 - (ii) The closed-loop step response is non-oscillatory and has a settling time of 2 seconds.
 - (iii) The DC gain of the transfer function from $e(s)$ to $y(s)$ is equal to 11.
-
- a) Draw the root locus of $G(s)$ accurately for all $K > 0$. [5]
 - b) Derive the location of the closed-loop pole that satisfies the second design specification. [5]
 - c) Show that the design specifications cannot be satisfied using a proportional compensator. [5]
 - d) Design a PD compensator that achieves the specifications. [5]
Hint: Define your compensator in terms of two parameters, say K_d and z . Next, obtain algebraic relations, perhaps involving the gain criterion, to satisfy the second and third specifications.
 - e) Draw the root locus of the compensated system. [5]
 - f) Evaluate the steady-state error of the closed-loop system for a unit step reference signal. [5]

3. Consider the feedback control system in Figure 3 below. Here,

$$G(s) = \frac{6}{(s+1)^3}$$

and $K(s)$ is the transfer function of a compensator.

- Sketch the Nyquist diagram of $G(s)$, indicating the low and high frequency portions. Also, calculate the real-axis intercepts. [7]
- Take $K = 1$. Show that the closed-loop is stable and determine the gain and phase margins. [7]
- Without doing any actual design, briefly describe how a phase-lead compensator,

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \quad 0 < \omega_0 < \omega_p,$$

would affect the gain and phase margins. Your answer should emphasize the difficulties involved in the design. [8]

- Design a stabilising phase-lead compensator $K(s)$ such that the loop gain has the same DC gain as $G(s)$ and the gain margin of $G(s)K(s)$ is infinite. Draw a rough sketch of the Nyquist diagram of $G(s)K(s)$. [8]

Hint: You may consider using a special type of phase-lead compensator that implements a pole-zero cancellation.

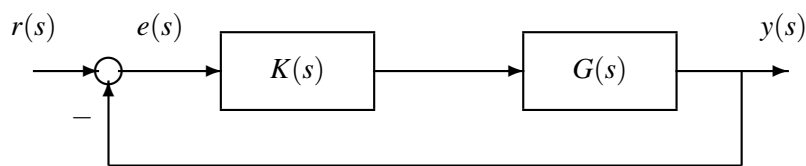


Figure 4

SOLUTIONS: Control Engineering 2010

1. a) i) Using the potential divider rule and the impedances we have

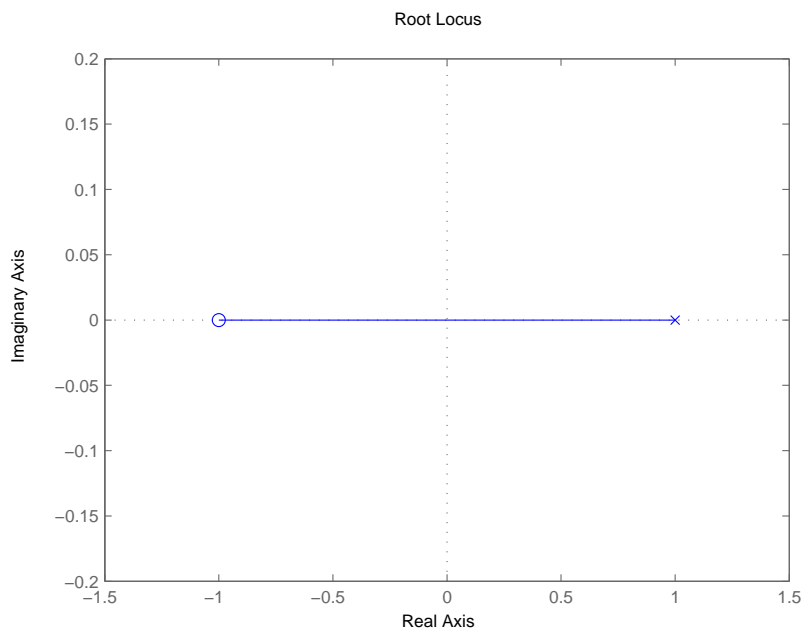
$$G(s) := \frac{v_o(s)}{v_i(s)} = \frac{sRC + 1}{s^2LC + sRC + 1}.$$

- ii) Using the final value theorem and the fact that $v_i(s) = 1/s$,

$$\lim_{t \rightarrow \infty} v_o(t) = \lim_{s \rightarrow 0} s v_o(s) = \lim_{s \rightarrow 0} s G(s) v_i(s) = \lim_{s \rightarrow 0} s G(s) \frac{1}{s} = G(0) = 1.$$

- iii) For marginal stability, the poles must be imaginary so $R = 0$. The frequency of oscillations is given by $\omega = \frac{1}{\sqrt{LC}}$.

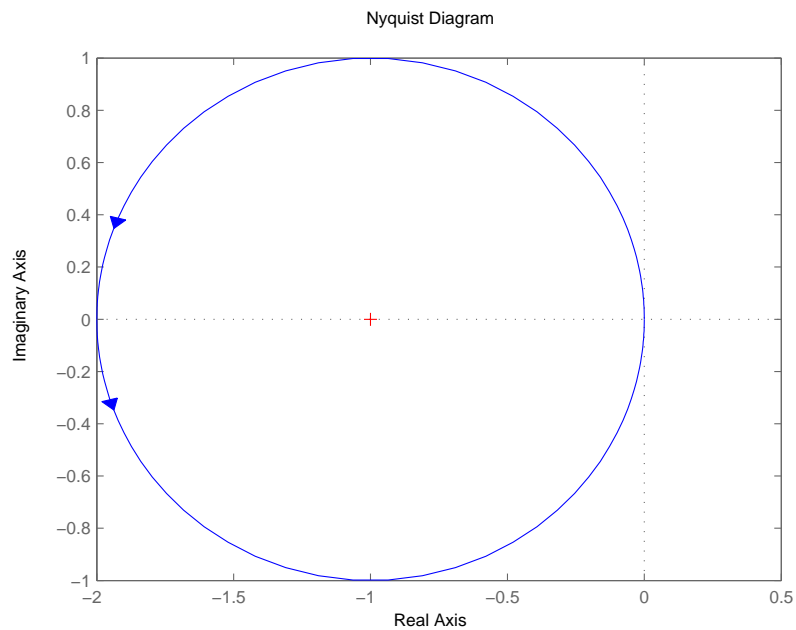
- b) i) The root-locus is shown below.



- ii) The closed-loop is marginally stable when at least one pole is on the imaginary axis and all others are in the left half-plane. It follows from the root-locus that the marginal pole is at $s = 0$. Using the gain criterion $K = -1/G(0) = 1$.

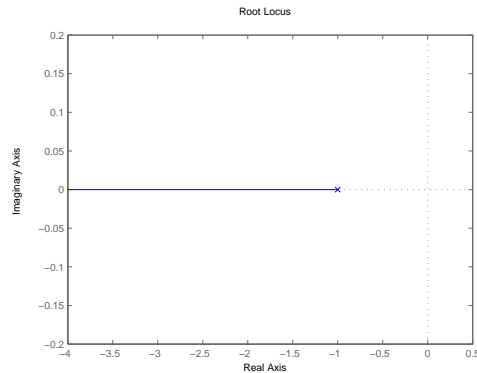
- iii) It follows from the root-locus that the closed-loop is stable for all $K > 1$.

- c) i) The Nyquist diagram is shown below. It is clear that the real axis intercepts are at -2 and 0 .

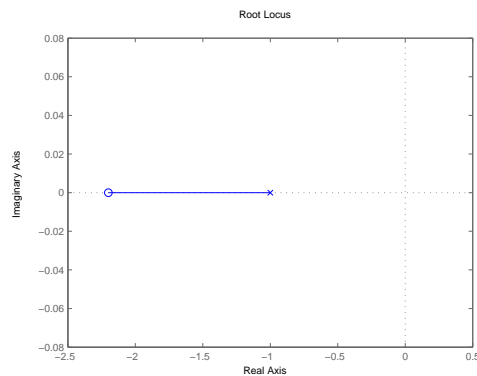


- ii) Let $K = 0.25$. The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by $G(s)$ of the point $-1/K$ as s traverses the Nyquist contour, which in this case is equal to 0; P is the number of unstable open-loop poles, which in this case is equal to 1; and Z is the number of unstable closed-loop poles. Thus there are $Z = N + P = 1$ unstable closed-loop poles.
- iii) When $K = 1$, then $N = -1$, $P = 1$ and so $Z = N + P = 0$ and the closed-loop is stable. Since the gain can be increased without bound the system has infinite gain margin for increasing gain. The gain can also be decreased by 50% before losing stability.

2. a) The root-locus is shown below.



- b) For a non-oscillatory response with a settling time of 2 seconds, the closed-loop pole must be located at -2 .
- c) Using the gain criterion, for a closed-loop pole at -2 , $K = -1/G(-2) = 1$. The resulting DC gain is then equal to $KG(0) = 1$ and the third specification is not specified. Thus there does not exist a proportional compensator that satisfies the design specifications.
- d) Following the hint, a PD compensator has the form $K(s) = K_d(s + z)$. To satisfy the second specification, the gain criterion requires that $1 - K_d(-2 + z) = 0$. To satisfy the DC gain criterion we need $K_d z = 11$. Therefore $K_d = 5$ and $z = 2.2$. So the compensator is $K(s) = 5(s + 2.2)$.
- e) The root-locus is shown below.



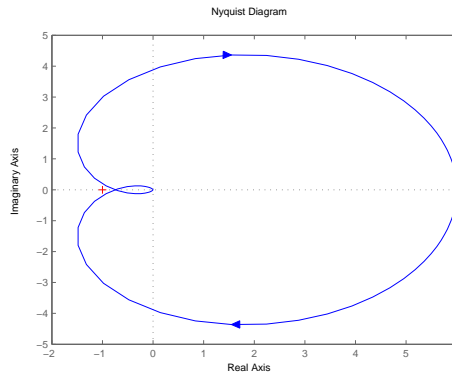
- f) The error signal is given by

$$e(s) = \frac{r(s)}{1 + G(s)K(s)}$$

Using the final value theorem for $r(s) = 1/s$ gives

$$e_{ss} = \frac{1}{1 + G(0)K(0)} = \frac{1}{12}$$

3. a) The Nyquist plot is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ to zero. This gives intercepts at $\omega_i = 0, \pm\sqrt{3}, \infty$ and so $G(j\omega_i) = 6, -0.75, -0.75, 0$.



- b) The number of unstable closed-loop poles is determined by the number of encirclements by $G(s)$ of the point -1 , which is zero. Thus the closed-loop is stable since $G(s)$ has no unstable poles. Since the real-axis intercept is at -0.75 , the gain margin is $4/3$. For the phase margin, we need the intercept with the unit circle centred on the origin. We solve $|G(j\omega)| = 1$, this gives $\omega_1 \sqrt{6^2 - 1}$ and $\arg [G(j\omega_1)] \approx -190^\circ$. The phase margin is then $\approx 10^\circ$.
- c) The phase-lead has gain close to 1 for $\omega < \omega_0$ and close to $\frac{\omega_p}{\omega_0} > 1$ for $\omega > \omega_p$. The phase is positive and large between ω_0 and ω_p but small elsewhere. Thus the gain increase for $\omega > \omega_p$ degrades stability margins while the phase-lead increases the phase margin. It is important to balance the destabilizing increase in gain and the stabilizing increase in phase. We should place w_p and w_0 in the crossover frequency range (when $|G(j\omega)| \approx 1$).
- d) One way of getting an infinite gain margin is to reduce the order of $G(s)$ from 3 to 2. This can be done using the PD compensator $K(s) = K_d(s + 1)$ (which is a special type of phase-lead compensator) since the zeros cancels one of the poles of $G(s)$. Taking $K_d = 1$ (to preserve the DC gain of $G(s)$), a sketch of the Nyquist diagram is given below.

