

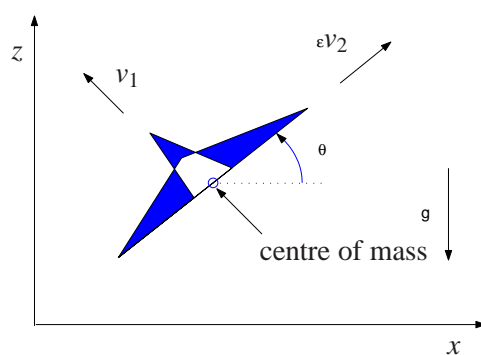
MODELLING AND CONTROL OF MULTIBODY MECHANICAL SYSTEMS

SAMPLE EXAM PAPER

1. Consider the (simplified and normalised) equations describing the motion of a vertical take-off and landing aircraft moving in a horizontal plane, namely

$$\begin{aligned}\ddot{x} &= -(\sin \theta)v_1 + \varepsilon(\cos \theta)v_2 \\ \ddot{z} &= (\cos \theta)v_1 + \varepsilon(\sin \theta)v_2 - g \\ \ddot{\theta} &= v_2,\end{aligned}$$

where (x, z) describes the position of the centre of mass of the aircraft in a vertical plane, θ the roll angle, g the gravity acceleration, v_1 and v_2 the control actions and $\varepsilon > 0$ a parameter that captures the effect of the “sloped” wings and induces a coupling between the vertical and the roll dynamics.



- a) Show that the system, with $v_1 = v_2 = 0$, can be written as an Hamiltonian system with $M = I$. In particular, write the internal Hamiltonian $H_0(q, p)$, where $q = (x, y, \theta)$ and p are the corresponding momenta. [4 marks]
- b) Verify that if $v_1 = v_2 = 0$ then $\dot{H}_0(q, p) = 0$. [2 marks]
- c) Show that the system is not a simple Hamiltonian system, *i.e.* it is not possible to define a Hamiltonian function $H(q, p, u) = H_0(q, p) - H_1(q)v_1 - H_2(q)v_2$, with $H_0(q, p)$ as in part a), such that

$$\dot{q} = \left(\frac{\partial H}{\partial p} \right)' \quad \dot{p} = - \left(\frac{\partial H}{\partial q} \right)'.$$

[4 marks]

- d) Let v_1 and v_2 be constant. Compute the equilibrium points of the system. Give a physical interpretation for the obtained result. [4 marks]
- e) Show that the point $(q, p) = (0, 0)$ is an equilibrium of the system. What is the value of the input signals associated to this equilibrium? [2 marks]
- f) Compute the linearization of the system around the equilibrium $(q, p) = (0, 0)$. Show that this equilibrium can be locally asymptotically stabilised by a state feedback control law exploiting Lyapunov first method. Finally, suppose that it is possible to measure only $y = q$. Is the equilibrium $(q, p) = (0, 0)$ locally asymptotically stabilizable by a dynamic output feedback control law? [4 marks]

2. Consider a simple Hamiltonian system with

$$H_0(q, p) = \frac{1}{2}p^2(1 + \alpha q^2) + \frac{1}{2}q^2 - \frac{1}{n}q^n,$$

$\alpha > 0$, $n > 2$ and even, and $H_1(q) = q$.

- a) Write the Hamiltonian equations of motion. [4 marks]
- b) Find the equilibria of the system for $u = 0$. [2 marks]
- c) Study the local stability properties of each equilibrium. [2 marks]
- d) Asymptotically stabilise the system around the equilibrium $(0, 0)$ using Lyapunov first method. [4 marks]
- e) Using the shaping function method find a control law which globally asymptotically stabilises the equilibrium $(0, 0)$. [8 marks]

3. A circular disk of mass m and radius a rolls without sliding on a horizontal plane as shown in Figure 3.1 (the disk in the figure has non-zero width for illustration purposes). The plane of the disk remains always vertical. The moment of inertia of the disk about its spin axis is I_{yy} and about a diameter is I_{zz} .

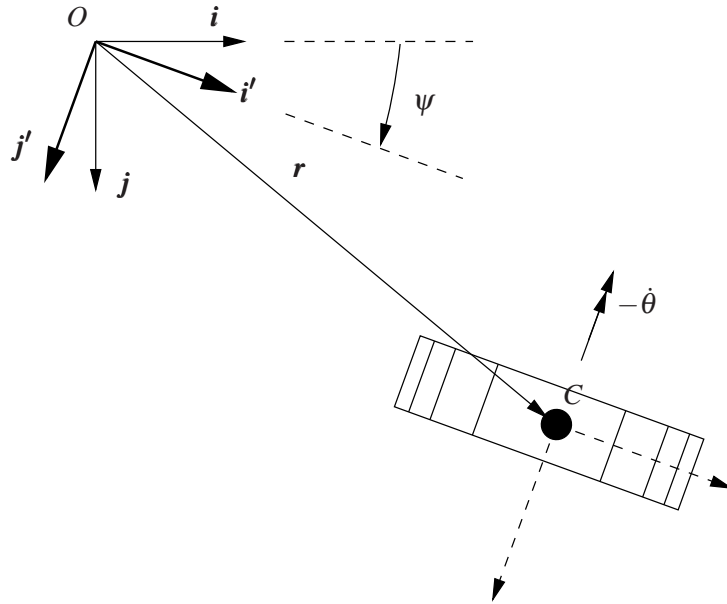


Figure 3.1 A disk rolling on a horizontal plane.

A moving Cartesian coordinate system with unit vectors i' and j' is used to analyse the motion of the object. This coordinate system has a fixed origin O but it rotates by an angle ψ so that it has the same orientation as the body fixed axes (shown with dashed lines on the object).

- The coordinates of the centre of mass, C , in the moving reference frame are (x', y') . Give the kinetic energy of the disk in terms of the four generalised coordinates x' , y' , ψ , θ . [5]
- Write the equations of the rolling constraint. [3]
- Hence derive the equations of motion of the object. [8]
- What are the forces that maintain the rolling constraint? [4]

4. a) A helicopter blade of mass m is attached onto a massless rotor that rotates with a fixed angular speed ω . The blade has a lagging freedom relative to the rotor described by the angle γ as shown in Figure 4.1.

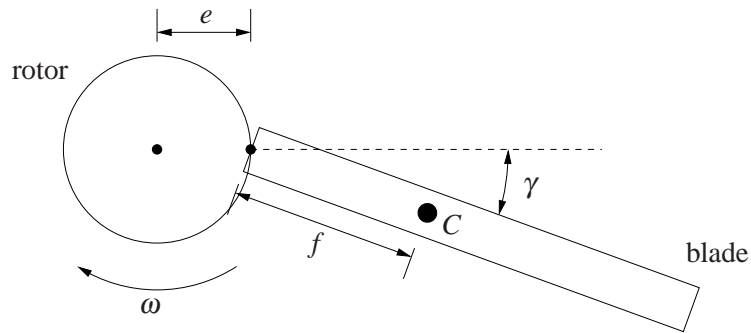


Figure 4.1 Plan view of a helicopter rotor with one blade.

The radius of the rotor is e , and the distance along the blade from the blade attachment point to the centre of mass of the blade, C , is f . The moment of inertia of the blade about the axis passing through the centre of mass and which is normal to the plane of the diagram is I_{zz} . A damping moment of magnitude $-D\dot{\gamma}$ opposes the motion of the blade relative to the rotor, where D is the damping coefficient.

- i) Write an expression for the kinetic energy of the system. [4]
 ii) Show that the lagging equation of motion is

$$(mf^2 + I_{zz})\ddot{\gamma} + D\dot{\gamma} + m\omega^2 ef \sin \gamma = 0.$$

[6]

- b) Prove that for a general rigid body motion about a fixed point the rate of change of the kinetic energy T is given by

$$\frac{dT}{dt} = \mathbf{\Omega} \cdot \mathbf{N},$$

where $\mathbf{\Omega}$ is the angular velocity vector of the body and \mathbf{N} is the external torque.

[10]

5. A cylindrical bar is made to rotate uniformly with an angular speed of ω about an axis passing through the centre of the bar and making an angle θ with the bar as shown in Figure 5.1.

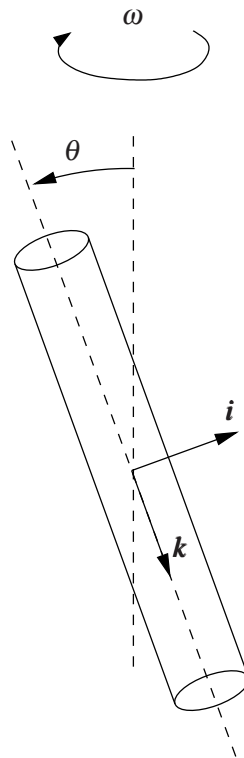


Figure 5.1 Cylindrical bar.

- a) Find the angular velocity vector of the bar expressed in a body fixed reference frame. [2]
- b) Find the angular momentum vector of the bar expressed in the same body fixed reference frame. [2]
- c) Find the magnitude and direction of the torque driving the bar. [6]
- d) If in addition to ω the bar rotates about its axis of symmetry with a speed of $\dot{\phi}$ then
 - i) what are the new angular velocity and angular momentum vectors? [5]
 - ii) what is the extra torque that is needed to drive the bar? [5]

6. A particle of mass m slides without friction on a wedge of angle α and mass M that can move without friction on a smooth horizontal surface, as shown in Figure 6.1.

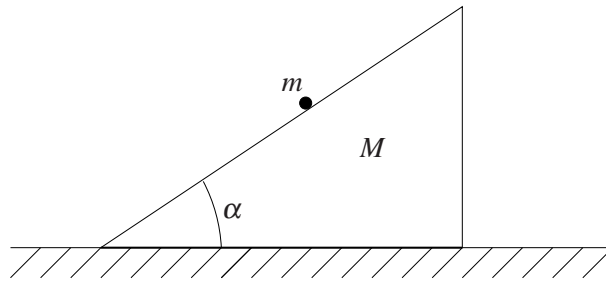


Figure 6.1 A particle slides on a wedge. The wedge slides on the horizontal surface.

- a) Treating the constraint of the particle on the wedge by the method of Lagrange multipliers, find the equations of motion for the particle and wedge. [15]
- b) Also obtain an expression for the forces of constraint between the particle and the wedge. [5]