QUESTION 1

a)
$$C(z) = \frac{1}{2}\left[H\left(z^{\frac{1}{2}}\right)X\left(z^{\frac{1}{2}}\right) + H\left(-z^{\frac{1}{2}}\right)X\left(-z^{\frac{1}{2}}\right)\right]$$

$$Y(z) = \frac{1}{2}G(z)\left[H(z)X(z) + H(-z)X(-z)\right]$$

$$D(z) = X(z) - Y(z)$$

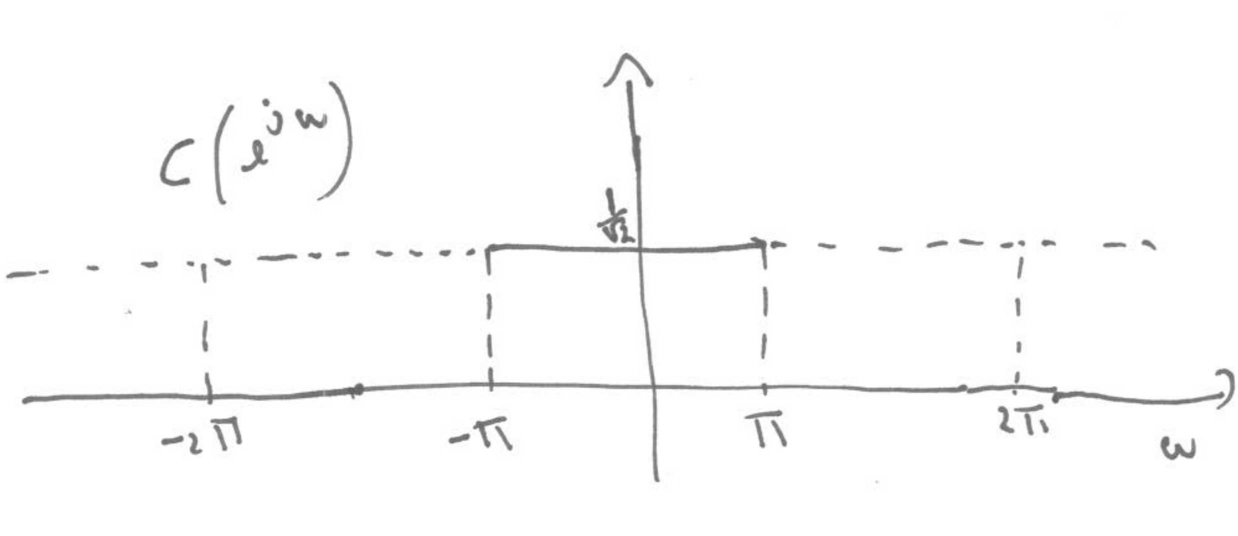
$$\hat{X}(z) = X(z) - \frac{1}{2}G(z)\left[H(z)X(z) + H(-z)X(-z)\right] + \frac{1}{2}F(z)\left[H(z)X(z) + H(-z)X(-z)\right]$$

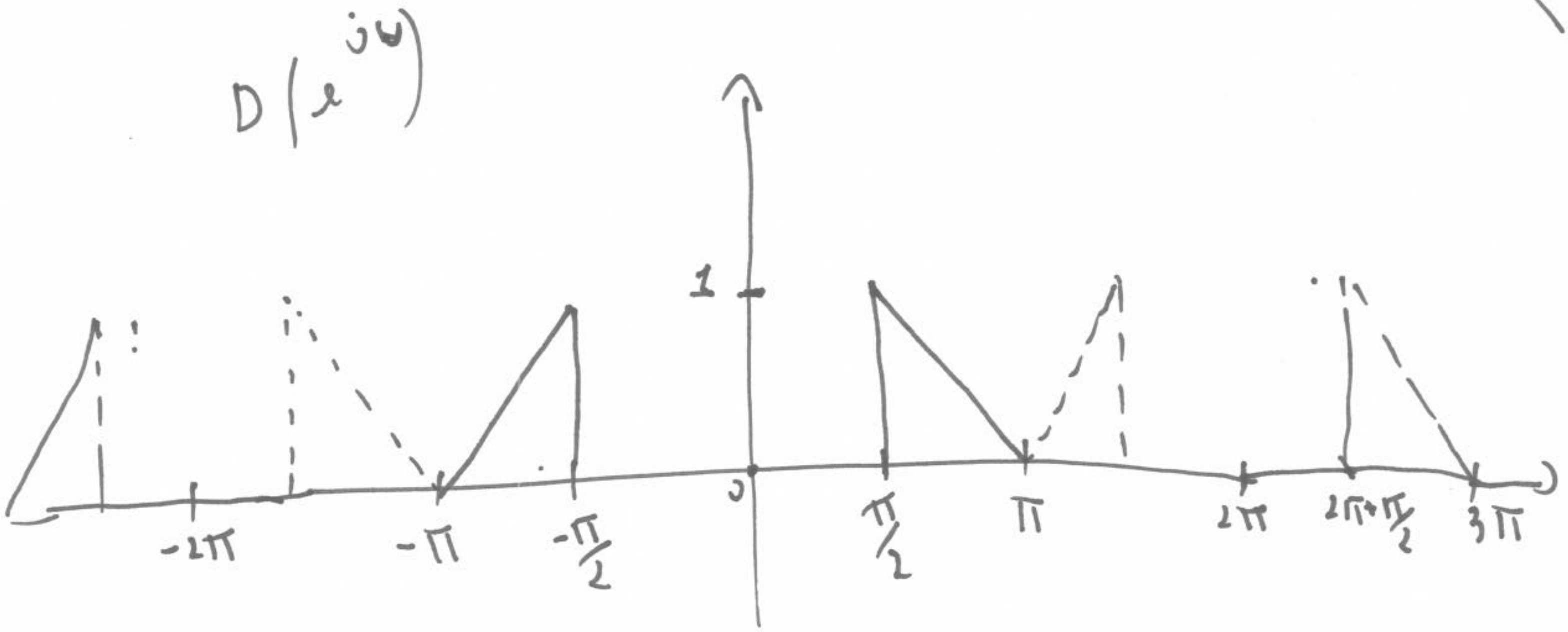
$$\Downarrow$$

PR: $F(z) = G(z)$

NOTICE THAT PR CONDITION DOES NOT DEPEND ON $H(z)$.

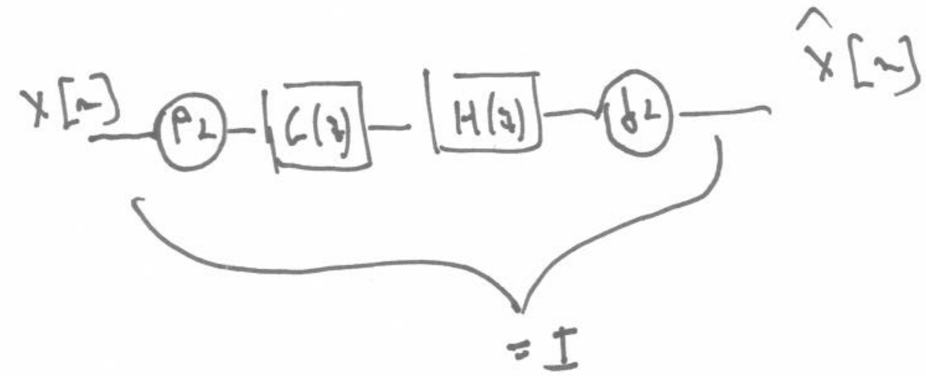
b)





c) THE SYSTEM IS NOT IDEMPOTENT : $P^2 \neq P$

IN FACT THE SYSTEM IS IDEMPOTENT IF



THAT IS $\hat{x}[n] = x[n]$

CHECK:

$$\hat{X}(z) = \downarrow H(z)\,G(z)\,X(z^2)\Big|_{\downarrow 2} =$$

$$= \frac{X(z)}{2}\left(H(z^{1/2})\,G(z^{1/2}) + H(-z^{1/2})\,G(-z^{1/2})\right)$$

$$= \frac{X(z)}{2} \underbrace{\left(z^{-1} + 6 + z\right)}_{\neq I}$$

d) THE SYSTEM IS IDEMPOTENT IF AND ONLY IF

$$H(z) \cdot G(z) + H(-z)\, G(-z) = 2 \qquad (1)$$

$$H(z) = (z+1)\left(z^{-1}+1\right)\left(a + bz + bz^{-1}\right)$$

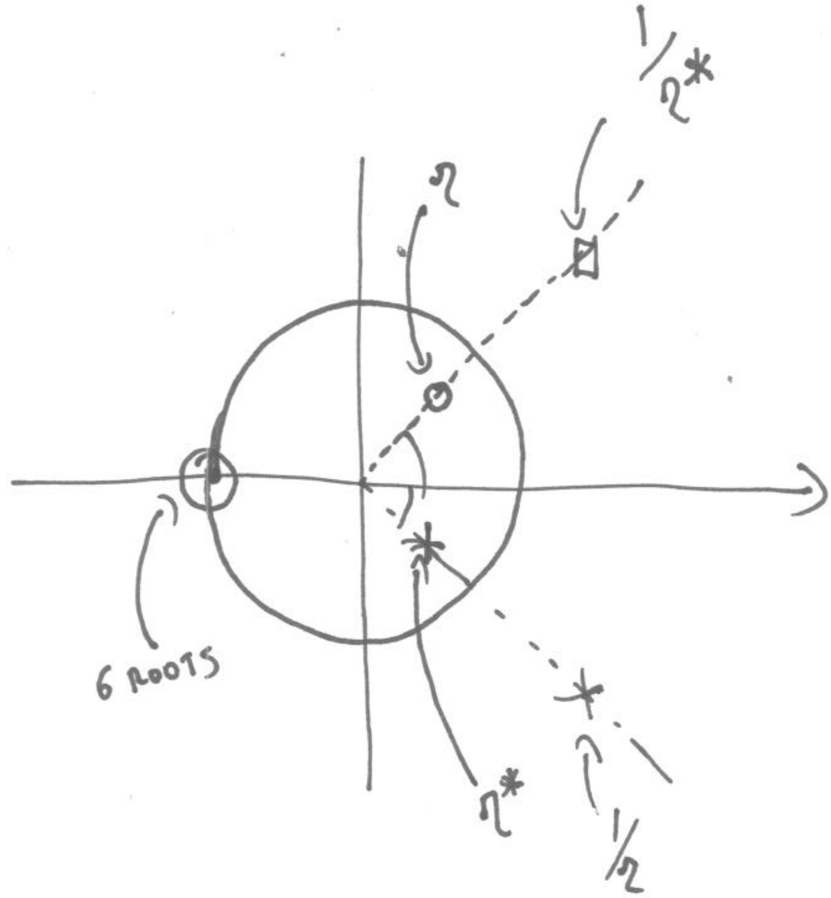
THE COEFFICIENTS a AND b MUST BE SUCH THAT CONDITION (1) IS SATISFIED.

THUS $a = \frac{1}{4}$ AND $b = -\frac{1}{16}$.

NOTICE THAT THE UPPER-BRANCH IS NOW PERFORMING AN OBLIQUE PROJECTION WHICH IS GOOD NEWS. HOWEVER, THE PROJECTION IS NOT OPTIMAL IN THAT IT IS NOT ORTHOGONAL.

QUESTION 2

a)



b) $$P(z) = (1+z)^3(1+z^{-1})^3(z-\zeta)(z-\zeta^{-1})(z-\zeta^*)\left(z-\frac{1}{\zeta^*}\right)$$

$$G_0(z) = (1+z^{-1})^3(z-\zeta)(z-\zeta^*)$$

$$H_0(z) = (1+z)^3\left(z-\frac{1}{\zeta}\right)\left(z-\frac{1}{\zeta^*}\right)$$

$$G_1(z) = -z^{-1}G_0(-z^{-1})$$

$$H_1(z) = G_1(z^{-1})$$

c) $$H_1(z) = \cancel{-z} \; -z(1-z^{-1})^3(z+\zeta)(z+\zeta^*)$$

THEREFORE $H_1(z)$ HAS A ZERO OF ORDER 3 ~~ROOT~~ AT $w=0$:

$$H_1(e^{jw})\Big|_{w=0}=0 \qquad H_1^{(1)}(e^{jw})\Big|_{w=0}=0 \qquad H_1^{(11)}(e^{jw})\Big|_{w=0}=0$$

NOW

$$\sum_k (m-k)^2 h_1[k] = m^2\sum_k h_1[k] - 2m\sum_k k\, h_1[k] + \sum_k k^2 h_1[k]$$

$$= 0 \quad \cancel{\text{SINCE}}$$

SINCE
$$\sum_k h_1[k] = H_1(e^{jw})\Big|_{w=0} = 0$$

$$\sum_k k\, h_1[k] = j\frac{dH_1}{dw}\Big|_{w=0} = 0$$

$$\sum_k k^2 h_1[k] = -\frac{d^2 H_1(e^{jw})}{dw}\Big|_{w=0} = 0$$

d) POSSIBLE FACTORIZATION:

$$G_0(z) = (1+z)^2(1+z^{-1})^2$$

$$H_0(z) = (1+z)(1+z^{-1})\Phi(z)$$

QUESTION 3

a)

TWO-SCALE EQUATIONS:

$$\varphi(t) = \sqrt{2} \sum_n g_0[n] \varphi(2t-n)$$

$$\tilde{\varphi}(t) = \sqrt{2} \sum_n h_0[n] \tilde{\varphi}(2t-n)$$

MOREOVER:

$$\langle \tilde{\varphi}(t), \varphi(t-n) \rangle = \delta[n] \qquad (1)$$

USING THE TWO-SCALE EQUATIONS AND THE LINEARITY OF INNER PRODUCT, WE OBTAIN:

$$\langle \tilde{\varphi}(t), \varphi(t-n) \rangle =$$

$$= 2 \sum_k h_0[k] \sum_\ell g_0[\ell] \langle \tilde{\varphi}(2t-k), \varphi(2t-2n-\ell) \rangle$$

$$= 2 \sum_k \sum_\ell h_0[k] g_0[\ell] \cdot \frac{1}{2} \langle \tilde{\varphi}(x-k), \varphi(x-2n-\ell) \rangle \qquad (2)$$

WHERE IN THE LAST EQUALITY WE HAVE REPLACED $2t$ WITH x.

BECAUSE OF (1) WE HAVE THAT

$$\langle \tilde{\varphi}(x-k), \varphi(x-2n-\ell) \rangle = \delta[k-(2n+\ell)]$$

EQ. 2, THEREFORE BECOMES:

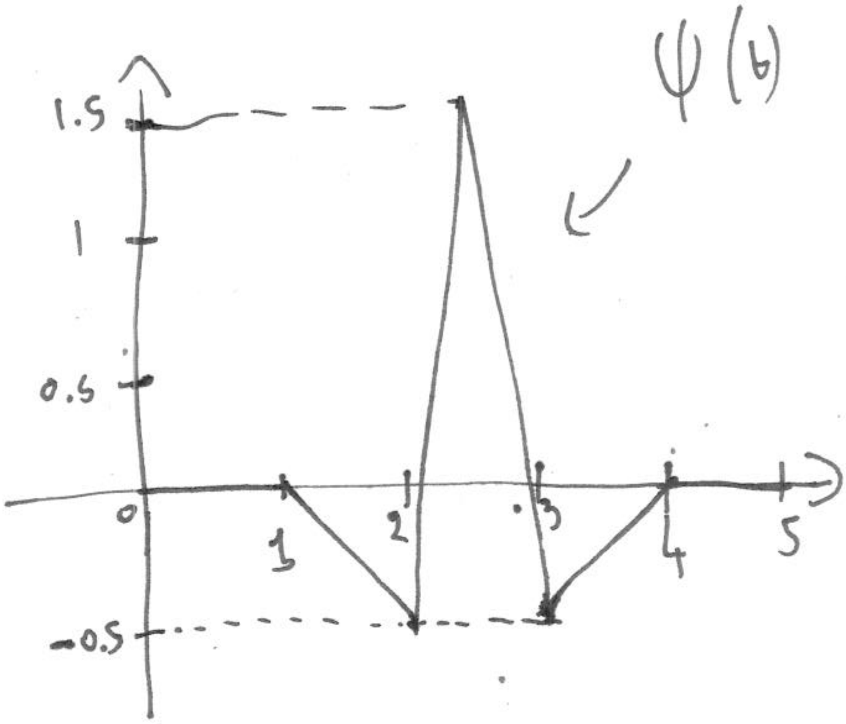
$$\langle \varphi^{s}(t), \varphi(t-n)\rangle = \cancel{1} \sum_{k}\sum_{l} h_0[k]\, g_0[l]\, \delta[k-(2n+l)]$$

$$= \cancel{\sum_{k}} \sum_{l} g_0[l]\, h_0[l+2n] = \langle g_0[l], h_0[l+2n]\rangle$$

BY COMPARING THIS LAST TERM ABOVE WITH EQ. (1) WE OBTAIN THE DESIRED RESULT:

$$\langle g_0[l], h_0[l+2n]\rangle = \delta[n]$$

b)



c) $\tilde{\psi}(t)$ HAS TWO VANISHING MOMENTS

IN FACT

$$\tilde{\psi}(t) = \sum_n \sqrt{2}\, h_1[n]\, \tilde{\varphi}(2t-n)$$

AND

$$h_1[n] = (-1)^{n-1} g_0[1-n]$$

SINCE $G_0(z)$ HAS TWO ZEROS AT π, THIS MEANS THAT $H_1(z)$ HAS TWO ZEROS AT $\omega=0$.

NOW,

$$\hat{\tilde{\psi}}(\omega) = \frac{1}{\sqrt{2}} H_1\left(e^{j\frac{\omega}{2}}\right) \hat{\tilde{\varphi}}\left(\frac{\omega}{2}\right)$$

THIS IMPLIES THAT

$$\hat{\tilde{\psi}}(\omega)\Big|_{\omega=0} = 0 \quad \text{AND THAT} \quad \frac{d\hat{\tilde{\psi}}(\omega)}{d\omega}\Big|_{\omega=0} = 0$$

THUS $\tilde{\psi}(t)$ HAS TWO VANISHING MOMENTS.

d) $f(t)$ IS α-LIPSHITS IN $t_0 \Rightarrow$

$$f(t) = p_{t_0}(t) + \varepsilon(t)$$

WHERE $p_{t_0}(t)$ IS A POLYNOMIAL OF DEGREE $m = \lfloor \alpha \rfloor$

AND $|\varepsilon(t)| \leq K |t - t_0|^{\alpha}$ (1)

THE WAVELET COEFFICIENTS IN THE CONE OF INFLUENCE OF t_0 ARE THEN GIVEN BY:

$$\langle f, \tilde{\psi}_{m,n} \rangle = \langle p_{t_0}(t), \tilde{\psi}_{m,n}(t) \rangle + \langle \varepsilon(t), \psi_{m,n} \rangle$$

SINCE $\alpha = 1.8$ THEN $m = 1$ AND
SINCE $\tilde{\psi}(t)$ HAS TWO VANISHING MOMENTS

$$\langle p_{t_0}(t), \tilde{\psi}_{m,n}(t) \rangle = 0$$

USING EQ. 1 WE HAVE:

$$\langle f, \tilde{\psi}_{m,n} \rangle = \langle \varepsilon(t), \tilde{\psi}_{m,n} \rangle \leq K 2^{-m/2} \int_{-\infty}^{\infty} |t - t_0|^{\alpha} \tilde{\psi}(2^{-m} t - n) dt$$

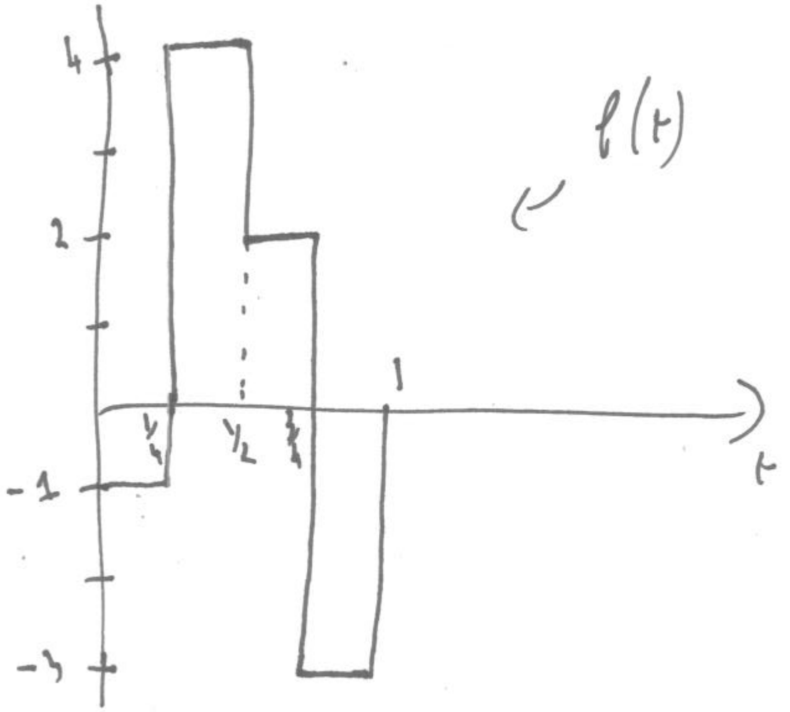
$$= K 2^{m/2} \int_{-\infty}^{\infty} |x 2^{m} + n 2^{m} - t_0|^{\alpha} \tilde{\psi}(x) dx$$

$$\overset{(a)}{\le} K C\, 2^{m(2+1/2)} \underbrace{\int_{-\infty}^{\infty} (|x|+|C|)^{2} \tilde{\psi}(x)\,dx}_{=A} =$$

$$= C_1\, 2^{m(2+1/2)}$$

WHERE (a) FOLLOWS FROM THE FACT THAT WE ARE IN THE CONE OF INFLUENCE OF t_0, THEREFORE $|n 2^{m} - t_0| \le C 2^{m}$. HERE 'C' IS THE COMPACT SUPPORT OF $\tilde{\psi}(t)$.

QUESTION 4



a) $\varphi_{-2,m}(t) = 2\varphi(4t-m)$, $c_{j,m} = \langle f(t), \varphi_{j,m}(t) \rangle$.

$$c_{-2,0} = -\frac{1}{2}$$

$$c_{-2,1} = 2$$

$$c_{-2,2} = 1$$

$$c_{-2,3} = -\frac{3}{2}$$

$$c_{-2,m} = 0 \qquad m \neq 0, 1, 2, 3$$

b) SINCE THE BASIS IS ORTHO-NORMAL, WE HAVE THAT

$$c_{j,m} = \langle f(t), \varphi_{j,m}(t) \rangle \quad \text{AND}$$

$$d_{j,m} = \langle f(t), \psi_{j,m}(t) \rangle$$

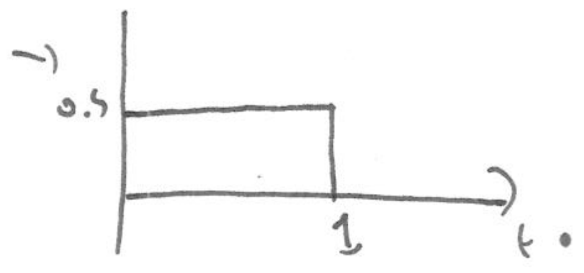
b)
$$\begin{cases} c_{0,0} = \frac{-1+4+2-3}{4} = \frac{1}{2} \\ c_{0,m} = 0 \quad m \neq 0 \end{cases}$$

$$\begin{cases} d_{0,0} = 1 \\ d_{0,m} = 0 \quad m \neq 0 \end{cases}$$

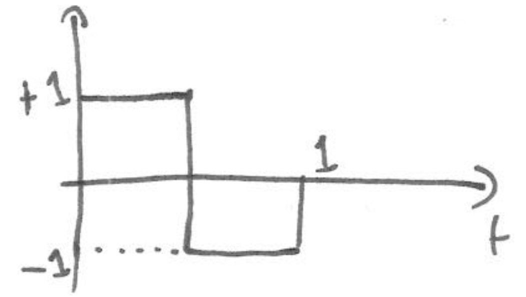
$$\begin{cases} d_{-1,0} = -\frac{5}{4} \cdot \sqrt{2} \\ d_{-1,1} = \frac{5}{4} \cdot \sqrt{2} \\ d_{-1,m} = 0 \quad m \neq 0,1 \end{cases}$$

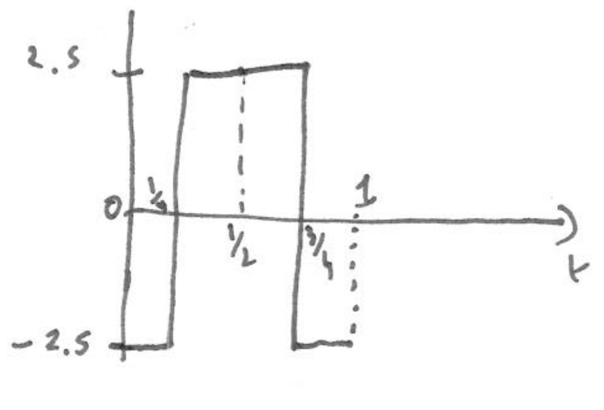
c)

$V_0 \rightarrow$



ψ_0





(13)

13

d)

PARSEVAL $\Rightarrow$ $\|f\|^2 = \sum_m |c_{m,0}|^2 + \sum_{j=-\infty}^{-1} \sum_m |d_{j,n}|^2$

$\|f\|^2 = \frac{1}{4} + \frac{16}{4} + \frac{4}{4} + \frac{9}{4} = \frac{15}{2}$

$|c_{0,0}|^2 + |d_{0,0}|^2 + |d_{-1,0}|^2 + |d_{-1,1}|^2 =$

$= \frac{1}{4} + 1 + \frac{25}{16}\cdot 2 + \frac{25}{16}\cdot 2 = \frac{15}{2}$ ✓