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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE PART IV: MEng and ACGI

Corrected Copy

ESTIMATION AND FAULT DETECTION

Wednesday, 21 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	R.B. Vinter
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Information for candidates:

Some formulae relevant to the questions.

The normal $N(m, \sigma^2)$ density:

$$p(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-m)^2}{2\sigma^2}\right)$$

System equations:

$$\begin{aligned} x_k &= Fx_{k-1} + u_k^s + w_k \\ y_k &= Hx_k + u_k^o + v_k \,. \end{aligned}$$

Here, w_k and v_k are white noise sequences with covariances Q^s and Q^0 respectively.

The Kalman filter equations are

$$\begin{aligned} P_{k|k-1} &= FP_{k-1}F^T + Q^s \\ P_k &= P_{k|k-1} - P_{k|k-1}H^T(HP_{k|k-1}H^T + Q^o)^{-1}HP_{k|k-1}\,, \\ K_k &= P_{k|k-1}H^T(HP_{k|k-1}H^T + Q^o)^{-1}\,, \\ \hat{x}_k &= \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1})\,, \end{aligned}$$

in which $\hat{x}_{k|k-1} = F\hat{x}_{k-1} + u_k^s$ and $\hat{y}_{k|k-1} = H\hat{x}_{k|k-1} + u_k^o$

1. Consider a stationary zero mean continuous 2-vector stochastic process $x(t) = [x^1(t), x^2(t)]^T$, governed by the stochastic differential equation

$$\frac{d}{dt}\begin{bmatrix} x^1(t) \\ x^2(t) \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}\begin{bmatrix} x^1(t) \\ x^2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

in which $w(t)$ is a stationary scalar white noise process with unit variance; i.e. $E[w(t)w(s)] = \delta(t-s)$. α_1 and α_2 are positive constants.

The discrete time process $x_k = [x_k^1, x_k^2]^T$ is obtained by sampling the continuous time process at times $\ldots, -2h, -h, 0, h, 2h, \ldots$; thus

$$x_k = x(kh).$$

(h, the sampling period, is a given positive constant.)

(a):) Show that x_k satisfies a stochastic difference equation of the form

$$x_k = Fx_{k-1} + v_k ,$$

in which v_k is a 2-vector white noise process with covariance Q. Evaluate F and Q. [8]

(b): Show that covariance R of x_k

$$R := \operatorname{cov}\{x_k\} \; (= E[x_k x_k^T]) .$$

satisfies the Lyapunov Equation

$$R = FRF^T + Q$$

[4]

Solve the Lyapunov equation for R. [4]

Why is R independent of h? [2]

(c): Show that, in the case when $\alpha_1/\alpha_2 > 100$ then the correlation coefficient

$$|\,\rho(x_k^1, x_k^2)\,| \leq 0.1$$

where $\rho(x_k^1, x_k^2)$ is the correlation coefficient of x_k^1 and x_k^2,

$$\rho(x_k^1, x_k^2) = \frac{E[\,x_k^1\, x_k^2\,]}{(E(x_k^1)^2)^{\frac{1}{2}}\,(E(x_k^2)^2)^{\frac{1}{2}}} ,$$

[2]

(This illustrates the fact that, if the time constants associated with the scalar processes x_k^1 and x_k^1 differ by an order of magnitude, then these processes are almost uncorrelated, even if they are generated by the same noise process.)

2. Denote the temperature of a reactor by x. A noisy measurement y is taken of x, using a thermometer. The thermometer is sometimes faulty; when it is faulty it introduces a bias of 1 unit in the measurement.

Model x as a normal random variable ($x \sim N(0, \sigma_x^2)$). Assume that y is governed by the equation

$$y = x + n + b \,,$$

in which the noise term n is a normal random variable ($n \sim N(0, \sigma_n^2)$) and the bias term b is a discrete random variable taking values 0 or 1, with probabilities

$$P[b=0] = (1-P) \quad \text{and} \quad P[b=1] = P.$$

Here, σ_x^2, σ_n^2 and P, $0 < P < 1$, are given positive constants. It is assumed that x ~~and~~ n and b are independent.

(a): Show that the linear least squares estimate $\hat{x}_L$ of x given y is of the form

$$\hat{x}_L = K(y - \alpha) \,,$$

and evaluate the constants K and α. [8]

(b): Show that the nonlinear least squares estimate x_N of x given y is of the form

$$\hat{x}_{NL} = K_1(y - \alpha(y)) \,,$$

for some constant K_1, where

$$\alpha(y) = \frac{P \times N(1, \sigma_x^2 + \sigma_n^2)(y)}{(1-P) \times N(0, \sigma_x^2 + \sigma_n^2)(y) + P \times N(1, \sigma_x^2 + \sigma_n^2)(y)}$$

and evaluate the constant K_1. [10]

Briefly comment on why $\hat{x}_{NL}$ and $\hat{x}_N$ differ. [2]

Hint: Use Bayes' rules to show that $\alpha(y) = P[b = 1 \,|\, y]$. Then note that $p(x \,|\, y)$ is a weighted sum of normal random variables given by

$$p(x \,|\, y) = p(x \,|\, y, b=0)p(b=0) \;+\; p(x \,|\, y, b=1)p(b=1)$$

4. Denote by x_k the n-dimensional state of a deterministic system, with random initial state x_0. Noisy measurements y_k of the state are taken at times $k = 1, 2, \ldots$ Assume that the evolution of the state and the measurement process are modelled by the equations

$$\begin{cases} x_k &= F x_{k-1} \\ y_k &= H x_k + v_k \, . \end{cases}$$

Here, F and H are given $n \times n$ and $r \times n$ dimensional matrices. $\{v_k\}$ is Gaussian white noise sequence, with covariance the given $k \times k$ matrix Q. x_0 is a normal random variable ($x_0 \sim N(\hat{x}_{0|0}, P_{0|0})$ for a given n-vector $\hat{x}_{0|0}$ and given $n \times n$ matrix $P_{0|0}$). It is assumed that x_0 and $\{v_k\}$ are independent.

A recursive filter is required, to estimate the value of the *initial* state x_0, based on measurement values $y_{0:k} := \{y_1, \ldots, y_k\}$, $k = 1, 2, \ldots$. Define the conditional means and covariances of x_0 given measurements up to time k:

$$\hat{x}_{0|k} = E[x_0 \,|\, y_{1:k}] \quad \text{and} \quad P_{0|k} = \text{cov}\{x_0 \,|\, y_{1:k}\} \, .$$

(a): Show that

$$p(y_k \,|\, y_{1:k-1}, x_0) = N(HF^k x_0, Q)(y_k) \, .$$

[4]

(b): Using Bayes' rule in the form

$$\log p(x_0 \,|\, y_{1:k}) \\ = \log p(y_k \,|\, y_{1:k-1}, x_0) + \log p(x_0 \,|\, y_{1:k-1}) - \log p(y_k \,|\, y_{1:k-1})$$

derive the recursive equations for $P_{0|k}$ and $\hat{x}_{0|k}$:

$$P_{0|k}^{-1} = P_{0|k-1}^{-1} + (F^k)^T H^T Q^{-1} H F^k$$

[6]

and

$$P_{0|k}^{-1} \hat{x}_{0|k} = P_{0|k-1}^{-1} \hat{x}_{0|k-1} + (F^k)^T H^T Q^{-1} y_k \, .$$

[7]

(c): Suppose that $n = r = 1$ (scalar state and observations). Suppose further that $P_{0|0} = 1$, $F = \sqrt{0.5}$ and $Q = 1$. Determine the limiting error covariance, $P_{0|\infty}$:

$$P_{0|\infty} := \lim_{k \to \infty} P_{0|k}.$$

[4]

5. Let X be a random variable with first and second moments

$$E[X] = 0 \qquad \text{and} \qquad E[X^2] = \sigma^2 \tag{1}$$

(for some known constant σ^2.) A noisy 'nonlinear' measurement Y is taken of X. Assume that Y is a random variable satisfying the equation

$$Y = h(X) + V ,$$

in which the noise term V is a zero mean random variable with known variance σ_n^2 and $h(x)$ is the cubic nonlinearity

$$h(x) = x^3 .$$

Construct a linear estimator for X given Y of the form

$$\hat{X} = KY$$

using the following method:

Step 1.

Assume that, for purposes of constructing the filter, X has a discrete distribution

$$p_X(x) = \alpha\delta(x+\sigma) + (1-2\alpha)\delta(x) + \alpha\delta(x-\sigma) ,$$

for some α, $0 < \alpha < 1$; otherwise expressed, X is assumed to be a discrete random variable taking values $-\sigma$, 0 or σ, with probability weights α, $(1-2\alpha)$ and α respectively. Determine the value of α such that X has the correct first two moments (see (1)). [4]

Step 2.

Calculate $E[XY]$ and $E[Y^2]$, using the probability distribution for X that you have just calculated. Choose the filter gain K to be the gain of the linear least squares estimator of X given Y. [10]

(This estimate is a version of the widely used 'unscented Kalman filter')

An alternative approach to constructing a linear estimator (*the 'extended Kalman filter'*) is to assume x is a Gaussian random variable and to approximate the nonlinear function $h(x)$ by the linear function $h(E[X]) + h_x(x - E[X])$, taking the correct value and slope at $x = E[X]$. Show that this gives an estimate:

$$\hat{x}_{EKS} = K_{EKS}Y$$

where [4]

$$H = h_x(E[X]) \quad \text{and} \quad K_{EKS} = \frac{\sigma^2 H}{H\sigma^2 H + \sigma_n} .$$

Why can we expect that the unscented Kalman filter performs far better than the extended Kalman filter in the case (1)? [2]

6. Consider the stationary, zero mean, Gaussian process y_k that satisfies the difference equation

$$y_k + 0.5y_{k-1} = e_k + de_{k-1} ,$$

in which e_k is a Gaussian white noise process with unit variance. d is a constant.

(a): Calculate the variance $r_d(0)$ of y_k:

$$r_d(0) = E[y_k^2]$$

(it will depend on the constant d). [10]

(b): Now suppose that the value of d depends on whether a fault has occurred. We consider two hypotheses:

(H_0): (a fault has not occurred) $d = 0$

(H_1): (a fault has occurred) $d = 2$.

Write $P_i[A]$, $i = 0, 1$, for the probability of the event A under hypotheses (H_0) and (H_1) respectively.

For a single value of k, a perfect measurement of $z = y_k$ is taken. Design a Neyman Pearson-type decision rule $\delta(z)$ that takes values 0 (no fault) and 1 (fault), and which maximizes the power of test, namely

$$P_1[\delta(z) = 1]$$

(the probability that the rule will detect a fault if it has occurred) at the 0.05 significance level, i.e. under the following constraint on the probability of a false alarm:

$$P_0[\delta(z) = 1] = 0.05 .$$

[6]

Determine the power of the test. [4]

You may use the following data about a normal random $x \sim N(0, 1)$:

c	:	0.039	0.01	1.28	3.84
$P[x^2 \geq c]$	:	0.95	0.74	0.26	0.05