Master - April 08

E4.25
C4.1
[ISE 4.2]

**Design of Linear Multivariable Control Systems**

**Solutions 2008**

1. (a) (i) Since $[A - sI \;\; B]$ loses rank for $s = -3$ and $s = -5$, they are uncontrollable modes, and since $[A^T - sI \;\; C^T]$ loses rank for $s = 4$ and $s = -5$, they are unobservable modes. Since the uncontrollable modes are stable, the realisation is stabilisable, and since one of the unobservable modes is unstable, the realisation is not detectable.

(ii) Since the modes $\lambda = -3$ and $\lambda = -5$ are uncontrollable, they cannot be assigned via state feedback and so they are eigenvalues of $A - BK$. Similarly, since $\lambda = 4$ and $\lambda = -5$ are unobservable modes, they cannot be assigned via output injection and so they are eigenvalues of $A - LC$.

(iii) By removing the uncontrollable and/or unobservable modes we get the minimal realisation

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|cc} 1 & 1 & 2 \\ \hline 2 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} \frac{s+1}{s-1} & \frac{4}{s-1} \\ \frac{1}{s-1} & \frac{s+1}{s-1} \end{bmatrix} = \frac{1}{s-1}\begin{bmatrix} s+1 & 4 \\ 1 & s+1 \end{bmatrix}.$$

(b) (i) The inequality implies that $A'P + PA < 0$. Let $z \neq 0$ be a right eigenvector of $A$ and let $\lambda$ be the corresponding eigenvalue. Then multiplying the inequality from the left by $z'$ and from the right by $z$ gives $(\lambda + \bar{\lambda})z'Pz < 0$. Since $P > 0$ and $z \neq 0$ then $z'Pz > 0$ and it follows that $\lambda + \bar{\lambda} < 0$ so that $A$ is stable.

(ii) Since $A$ is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0) = 0$,

$$J := \int_0^\infty [y'y - \gamma^2 u'u] dt < 0,$$

for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now,

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0.$$

So,

$$\begin{aligned} 0 &= \int_0^\infty \dot{x}'Px + x'P\dot{x}dt = \int_0^\infty [(Ax+Bu)'Px + x'P(Ax+Bu)]dt \\ &= \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt. \end{aligned}$$

Use $y = Cx$ and add the last expression to $J$ and using the hint,

$$\begin{aligned} J &= \int_0^\infty [x'(A'P+PA+C'C)x + x'PBu + u'B'Px - \gamma^2 u'u]dt \\ &= \int_0^\infty [x'(A'P+PA+C'C + \gamma^{-2}PBB'P)x - \|(\gamma u - \gamma^{-1}B'Px)\|^2]dt < 0 \end{aligned}$$

from the inequality. It follows that $\|H\|_\infty < \gamma$. Comparing with the inequality, it follows that $\gamma = 2$.

2. (a) Inject a signal $d$ in between $G(s)$ and $K(s)$ and call the input to $G(s)$ $u$. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} d \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

the loop is internally stable if and only if $T^{-1}(s)$ is stable.

(b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}.$$

Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I-GK)^{-1} \\ 0 & (I-GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since $(I - GK)^{-1} = I + GK(I - GK)^{-1}$, it follows that if $G$ is stable, then the loop is internally stable if and only if $Q := K(I - GK)^{-1}$ is stable. Rearranging terms shows that $K$ internally stabilising if and only if $K = Q(I + GQ)^{-1}$ for some stable $Q$.

(c) Since $K$ is required to be internally stabilising, $K = Q(I + GQ)^{-1}$ for some stable $Q$ from Part (b). We search for a stable $Q$ to satisfy the requirements.

i Since the transfer matrix from $r$ to $e$ is

$$S(s) = (I - G(s)K(s))^{-1} = I + G(s)Q(s)$$

we need

$$\|I + GQ\|_\infty < 1.$$

ii Let the input to $\Delta$ be $\epsilon$ while the output from $\Delta$ be $\delta$. Then $\epsilon = C\delta$ where $C = (I - GK)^{-1}GK$ which is stable. Now $C = GK(I - GK)^{-1} = GQ$. The small gain theorem implies that for $K$ to stabilise the loop in Figure 2.2 for all $\Delta$ such that $\|\Delta\|_\infty < 1$, we must have

$$\|GQ\|_\infty < 1.$$

Since $G(s)$ is minimum phase $G(s)^{-1}$ is stable and we set $Q(s) = \alpha G(s)$ and choose $\alpha$ to satisfy the design specifications. The specification in (i) requires

$$|1 + \alpha| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < 1 + \alpha < \frac{1}{2} \Leftrightarrow -\frac{3}{2} < \alpha < -\frac{1}{2}.$$

The second specification requires that

$$|\alpha| < 1 \Leftrightarrow -1 < \alpha < 1.$$

Combining these specifications, a family of internally stabilising controllers that achieves the design specifications is given by $K = Q(I + GQ)^{-1}$ where $Q = \alpha G(s)^{-1}$ and where $-1 < \alpha < -\frac{1}{2}$. That is, $K = \frac{\alpha}{1+\alpha} G(s)^{-1}$.

For the last part, since $KS = Q = \alpha G^{-1}$, it follows that the smallest achievable $\|KS\|_\infty$ is $0.5\|G^{-1}\|_\infty$.

3. (a) Let $V = x^T P x$ and set $u = Fx$. Provided that $P = P^T > 0$ and $\dot{V} < 0$ along closed-loop trajectories, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T \left(A^T P + PA + F^T B^T P + PBF\right) x.$$

Integrating from 0 to $\infty$ and using $x(\infty) = 0$,

$$\int_0^\infty x^T \left(A^T P + PA + F^T B^T P + PBF\right) x dt = -x_0^T P x_0.$$

(b) Using the definition of $J$ and adding the last equation,

$$J = x_0^T P x_0 + \int_0^\infty x^T [A^T P + PA + I + F^T F + F^T B^T P + PBF] x dt.$$

Completing the square using $(F + B^T P)^T (F + B^T P) = F^T F + F^T B^T P + PBF + PBB^T P$ gives $J = x_0^T P x_0 + \int_0^\infty \{x^T [A^T P + PA + I - PBB^T P] x + \left\|(F + B^T P) x\right\|^2\} dt$. Since the last term is always nonnegative, it follows that the minimizing value of $F$ is given by $F = -B^T P$. We can set the term in square brackets to zero provided $P$ satisfies the Riccati equation,

$$A^T P + PA + I - PBB^T P = 0.$$

It follows that the minimum value of $J$ is $x_0^T P x_0$.

(c) For closed loop stability we need to prove that $A_c := A - BB'P$ is stable. The Riccati equation can be written as $A_c' P + PA_c + I + PBB'P = 0$. Let $\lambda \in \mathcal{C}$ be an eigenvalue of $A_c$ and $z \neq 0$ be the corresponding eigenvector. Pre– and post–multiplying the Riccati equation be $z'$ and $z$ respectively gives $(\lambda + \bar{\lambda}) z' P z + z' z + z' PBB' P z = 0$. Since $P > 0$ and $z \neq 0$, $z' P z > 0$, $z' z > 0$ and $z' PBB' \geq 0$. It follows that $\lambda + \bar{\lambda} < 0$ and the closed loop is stable.

(d) By direct evaluation, $L(j\omega)' L(j\omega) =$

$$I - F(j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} F' + B'(-j\omega I - A')^{-1} F' F (j\omega I - A)^{-1} B$$

But $F'F = A'P + PA + I = -(-j\omega I - A')P - P(j\omega I - A) + I$ from the Riccati equation. So, $L(j\omega)' L(j\omega)$

$$\begin{aligned} &= I - F(j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} F' \\ &\quad + B'(-j\omega I - A')^{-1} [-(-j\omega I - A')P - P(j\omega I - A) + I](j\omega I - A)^{-1} B \\ &= I - [F + B'P](j\omega I - A)^{-1} B - B'(-j\omega I - A')^{-1} [F' + PB] \\ &\quad + B'(-j\omega I - A')^{-1} (j\omega I - A)^{-1} B = I + G(j\omega)' G(j\omega) \end{aligned}$$

(e) Let $\epsilon$ be the input to $\Delta$ and $\delta$ be the output of $\Delta$. Then $\epsilon = \delta + FG\epsilon = (I - FG)^{-1} \delta$. Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta (I - FG)^{-1}\|_\infty < 1$. But Part (d) implies that $\underline{\sigma}[I - FG(j\omega)] \geq 1 \forall \omega$ which implies $\|(I - FG)^{-1}\|_\infty \leq 1$. This shows that the loop will tolerate perturbations $\Delta$ of size $\|\Delta\| < 1$ without losing internal stability.

4. (a) It is clear that we require $K$ to internally stabilize the nominal model.

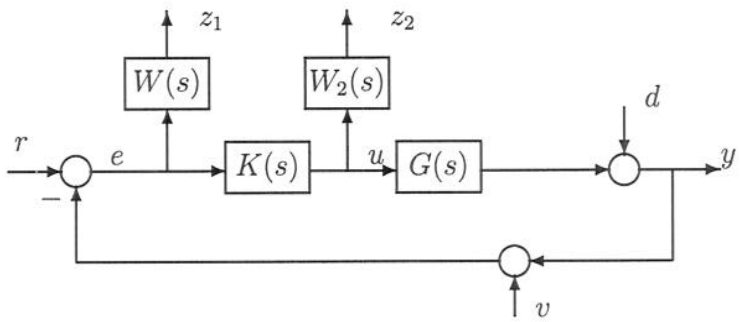
(i) Suppose that $\Delta_1 = 0$ and let the input to $\Delta_2$ be $\epsilon_2$ while the output from $\Delta_2$ be $\delta_2$. Then a calculation shows that $\epsilon_2 = -KS\delta_2$ where $S = (I+GK)^{-1}$ is the sensitivity which is stable. Using the small gain theorem, to satisfy the first design requirement, it is sufficient that $\|\Delta_2(j\omega)K(j\omega)S(j\omega)\| < 1, \forall\omega$. This can be satisfied if $\|W_2KS\|_\infty < 1$, where $W_2 = w_2^{-1}I$.

(ii) An analogous procedure shows that to satisfy the second design requirement, it is sufficient that $\|\Delta_1(j\omega)S(j\omega)\| < 1, \forall\omega$. This can be satisfied if $\|W_1S\|_\infty < 1$, where $W_1 = w_1^{-1}I$.

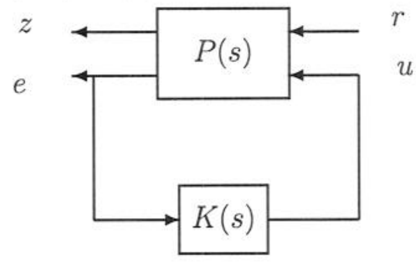
(iii) For the nominal loop, $y_o = (I+GK)^{-1}GKr$ so that $(I+GK)y_o = GKr$. For the loop with $\Delta_2 = 0$, $y_1 = (I+(I+\Delta_1)^{-1}GK)^{-1}(I+\Delta_1)^{-1}GKr$ so that $(I+(I+\Delta_1)^{-1}GK)y_1 = (I+\Delta_1)^{-1}GKr$. Substituting $(I+GK)y_o = GKr$ and multiplying from the left by $(I+\Delta_1)$ gives $(I+\Delta_1+GK)y_1 = (I+GK)y_o$ and so $(I+GK)(y_o-y_1) = \Delta_1 y_1$ or $y_o - y_1 = S\Delta_1 y_1$. Thus to satisfy the robust tracking requirement, it is sufficient that $\|\epsilon^{-1}W_1S\|_\infty < 1$.

We can combine the second and third requirements as $\|WS\|_\infty < 1$ where $W = W_1/min(1,\epsilon)$. To satisfy all three design requirements, it is sufficient that $\left\| \begin{bmatrix} WS \\ W_2KS \end{bmatrix} \right\|_\infty < 1$.

(b) The design specifications reduce to the requirement that the transfer matrix from $r$ to $z = [z_1^T \;\; z_2^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.




The corresponding generalized regulator formulation is to find an internally stabilizing $K$ such that $\|\mathcal{F}_l(P,K)\| < 1$:




$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} W & -WG \\ 0 & W_2 \\ \hline I & -G \end{array}\right].$$

5. (a) (i) The $(1,1)$ block of the inequality gives the inequality $A'P + PA < 0$. Let $z \neq 0$ be a right eigenvector of $A$ and let $\lambda$ be the corresponding eigenvalue. Then multiplying the inequality from the left by $z'$ and from the right by $z$ gives $(\lambda + \bar{\lambda})z'Pz < 0$. Since $P > 0$ it follows that $z'Pz > 0$ and it follows that $\lambda + \bar{\lambda} < 0$ so that $A$ is stable.

(ii) Since $A$ is stable, $\|H\|_\infty < \gamma$ if and only if, with $x(0) = 0$, $J := \int_0^\infty [y'y - \gamma^2 u'u]dt < 0$, for all $u(t)$ such that $\|u\|_2 < \infty$. If $\|u\|_2$ is bounded, then $\lim_{t\to\infty} x(t) = 0$. Now, $\int_0^\infty \frac{\mathrm{d}}{\mathrm{dt}}[x'Px]dt = x(\infty)'Px(\infty) - x(0)'Px(0) = 0$. So,

$$0 = \int_0^\infty (\dot{x}'Px + x'P\dot{x})\, dt = \int_0^\infty [x'(A'P+PA)x + x'PBu + u'B'Px]dt.$$

Use $y = Cx + Du$ and add the last expression to $J$

$$J = \int_0^\infty [x'(A'P+PA+C'C)x + 2x'(PB+C'D)u + u'(D'D-\gamma^2 I)u]dt$$

$$= \int_0^\infty \begin{bmatrix} x' & u' \end{bmatrix} \overbrace{\begin{bmatrix} A'P+PA+C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix}}^{M} \begin{bmatrix} x \\ u \end{bmatrix} dt.$$

It follows that $J < 0$, and so $\|H\|_\infty < \gamma$, if $M < 0$. This proves the result.

(b) (i) Substituting $u = Ly$, $y = Cx + w_2$ into the state equation gives

$$\dot{x} = \underbrace{(A + LC)}_{A_c} x + \underbrace{\begin{bmatrix} B & L \end{bmatrix}}_{B_c} w, \qquad y = \underbrace{C}_{C_c} x + \underbrace{\begin{bmatrix} 0 & I \end{bmatrix}}_{D_c} w.$$

It follows that $T_{yw}(s) = D_c + C_c(sI - A_c)^{-1}B_c$.

(ii) Using the results of part (a), by replacing $A$, $B$, $C$ and $D$ by $A_c$, $B$, $C$ and $D$, we have that there exists a feasible $L$ if there exists $P = P^T > 0$ such that

$$\begin{bmatrix} (A + LC)'P + P(A + LC) + C'C & PB & PL + C' \\ B'P & -\gamma I & 0 \\ L'P + C & 0 & -(\gamma - 1)I \end{bmatrix} < 0.$$

Noting that the only nonlinearity is due to the product $PL$, we define $Z = PL$ and so there exists a feasible $L$ if there exists $P = P^T > 0$ and $Z$ such that

$$\begin{bmatrix} A'P + PA + ZC + C'Z' + C'C & PB & Z + C' \\ B'P & -\gamma I & 0 \\ Z' + C & 0 & -(\gamma - 1)I \end{bmatrix} < 0.$$

6. (a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \ u(s) = Fy(s), P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline\hline C & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

(b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T > 0$ and $\dot{V} < 0$ along the closed-loop trajectory, we can assume $\lim_{t \to \infty} x(t) = 0$. Then

$$\dot{V} = \dot{x}^T X x + x^T X \dot{x} = x^T \left( A^T X + XA + F^T B^T X + XBF \right) x + x^T X B w + w^T B^T X x.$$

Integrating from 0 to $\infty$ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T \left( A^T X + XA + F^T B^T X + XBF \right) x + x^T X B w + w^T B^T X x] dt.$$

Using the definition of $J$ and adding the last equation, $J =$

$$\int_0^\infty \{x^T [A^T X + XA + C^T C + F^T F + F^T B^T X + XBF] x - [\beta w^T w - x^T Z^T w - w^T Z x]\} dt$$

where $Z = F + B^T X$ and $\beta = \gamma^2 - 1 > 0$ since $\gamma > 1$ by assumption. Completing the squares by using

$$Z^T Z = F^T F + F^T B^T X + XBF + XBB^T X$$
$$\|(\sqrt{\beta} w - \sqrt{\beta^{-1}} B^T X x)\|^2 = \beta w^T w - w^T B^T X x - x^T X B w + \beta^{-1} x^T X B B^T X x,$$

$$J = \int_0^\infty \{x^T [A^T X + XA + C^T C - XBB^T X] x + (1 + \beta^{-1}) \|Zx\|^2 - \left\| \sqrt{\beta} w - \sqrt{\beta^{-1}} Z x \right\|^2 \} dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of $X$ such that

$$A^T X + XA + C^T C - XBB^T X = 0, \qquad X = X^T > 0.$$

The feedback gain is obtained by setting $Z = 0$ so $F = -B^T X$. The worst case disturbance is $w^* = \beta^{-1} Z x = 0$. The closed-loop with $u = Fx$ and $w = w^*$ is $\dot{x} = [A - BB^T X] x$ and a third condition is $Re\ \lambda_i [A - BB^T X] < 0$, $\forall\, i$. It remains to prove $\dot{V} < 0$ for $u = Fx$ and $w = 0$. But

$$\dot{V} = x^T \left( A^T X + XA + F^T B^T X + XBF \right) x = -x^T (C^T C + XBB^T X) x < 0$$

for all $x \neq 0$ (since $(A, B, C)$ is assumed minimal) proving closed-loop stability.

(c) It is clear that our procedure breaks down if $\gamma \leq 1$ since in that case $\beta \leq 0$. Thus the smallest value of $\gamma$ is 1.