Stability and control of nonlinear systems: Model Answers 2008

1. (a) {*unseen*}

(i) The general case of these systems is $\ddot{x}+\kappa x = d$ with $\kappa > 0$ (#), for which the solution has the form $x(t) = \alpha+\beta\cos(\omega t+\phi)$ with $\dot{x}(t) = -\beta\omega\sin(\omega t+\phi)$. Substituting this into (#) gives the values $\alpha = d/\kappa$, $\omega = \sqrt{k}$ with β and ϕ determined by the initial conditions. For f_1 we obtain $\alpha = 1, \omega = \sqrt{k}$ corresponding to a circular trajectory centred on $(1, 0)$ and with radius β. For f_2 we have $\alpha = 0$ and $\omega = 1$, corresponding to circles centred on $(0, 0)$ with radius β (in general not the same as the previous β). Hence we obtain the trajectories shown in Figure A1.1. **[3]**

(ii) Consider $\ddot{x} = f_3(x, \dot{x}) = -\dot{x}$, i.e. $\frac{d\dot{x}}{dx}\,\dot{x} = -\dot{x}$ giving $\frac{d\dot{x}}{dx} = -1$ or $\dot{x} = 0$. Similarly for $\ddot{x} = f_4(x, \dot{x}) = \dot{x}$, we have $\frac{d\dot{x}}{dx} = 1$ or $\dot{x} = 0$. Hence we obtain the trajectories shown in Figures A1.2-3. **[3]**

(iii) By making use of the trajectories of Figures A1.1-3 we obtain the trajectory of Figure A1.4. **[4]**
The time taken for $\dot{x}$ to decrease from 3 to 2, corresponding to x moving from 0 to 1, is $\int_0^1 \frac{1}{\dot{x}}dx =$ (from Figure A1.1) $\int_0^1 \frac{1}{3-x}dx = -\ln(3-x)|_0^1 = \ln(3)-\ln(2)$ s. The time taken to move from $(1, 2)$ to $(1,-2)$ is $\pi/\omega = \pi$. Hence the total time needed is $\pi+\ln(3) - \ln(2)$ s. **[3]**

(b) {*bookwork*}

(i) The Fourier series for u is $u(t) = \sum_{k=0}^{\infty} a_k(a)\sin(k\omega t) + \sum_{k=0}^{\infty} b_k(a)\cos(k\omega t)$

$\approx b_0(a) + a_1(a)\sin(\omega t) + b_1(a)\cos(\omega t)$ (keeping only the contributions at freq. ω)
where $a_1(a) = \frac{\omega}{\pi}\int_0^T n(a\sin(\omega t))\sin(\omega t)\,dt$, $b_1(a) = \frac{\omega}{\pi}\int_0^T n(a\sin(\omega t))\cos(\omega t)\,dt$ and $T = \frac{2\pi}{\omega}$. Hence, and since the skew-symmetry of n yields $b_0(a) = 0$, we obtain $u(t) \approx a_1(a)\sin(\omega t) + b_1(a)\cos(\omega t) = \sqrt{a_1(a)^2+b_1(a)^2}\,\sin(\omega t+\phi)$ where $\phi = \text{atan}(\frac{b_1(a)}{a_1(a)})$. So the $\overline{a}$ required is $\sqrt{a_1(a)^2+b_1(a)^2}$. Further, $a_1(a)$ and $b_1(a)$ are independent of ω.

Hence the result of n operating on e can be approximated by the describing function $N(a) = \frac{\sqrt{a_1(a)^2+b_1(a)^2}}{a}e^{j\psi} = \frac{a_1(a)}{a}+j\frac{b_1(a)}{a}$. **[4]**

(ii) The harmonic balance equation is $1 = N(a)g(j\omega)$ i.e. $g(j\omega) = -\frac{1}{N(a)}$. Hence if the locus of $g(j\omega)$, as ω varies, and the locus of $-\frac{1}{N(a)}$, as a varies, intersect then an oscillation is predicted with the amplitude a and frequency ω corresponding to the intersection point. See Figure A1.5. **[3]**

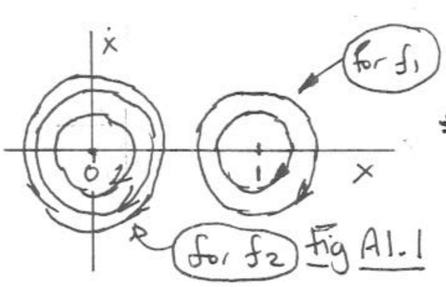

Fig A1.1

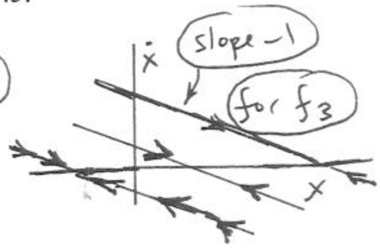

Fig. A1.2

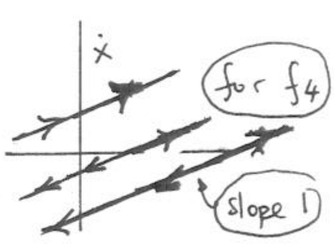

Fig A1.3

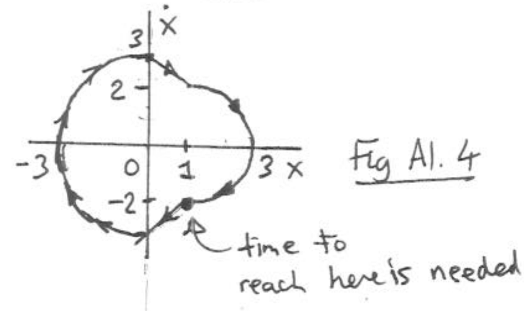

Fig A1. 4

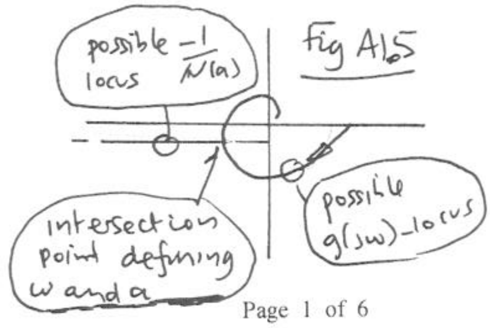

Fig A1.5

2. (a) {*bookwork*}

Now for all $x \in \mathbb{R}^n$, $\lambda_{\min}(P)\|x\|^2 \le x^{\mathrm{T}}Px = v(x,t)$ where $\lambda_{\min}(P)$ is the smallest eigenvalue of P and is strictly positive since $P > 0$. Hence $\psi(\|x\|) \triangleq \lambda_{\min}(P)\|x\|^2$ is a class-K function that satisfies $\psi(\|x\|) \le v(x,t)$ for all $x \in \mathbb{R}^n$ and all t. In addition, $\psi(\|x\|) \to \infty$ as $\|x\| \to \infty$. Hence $v(x,t)$ is radially-unbounded positive-definite on $\mathbb{R}^n$. Similarly, $x^{\mathrm{T}}Px \le \lambda_{\max}(P)\|x\|^2$ so we can define $\phi(\|x\|) = \lambda_{\max}(P)\|x\|^2$ where $0 < \lambda_{\max}(P) < \infty$.Then $v(x,t) \le \phi(\|x\|)$ for all $x \in \mathbb{R}^n$ and all t so $v(x,t)$ is decrescent on $\mathbb{R}^n$.

[4]

(b) {*unseen examples*}

(i) The origin is an equilibrium state of (2.1) since if we regard (2.1) as $\dot{x} = f(x,t)$ then $f(0,t) = 0$ for all t.
Since $x_1^2 + 2x_2^2 = x^{\mathrm{T}}Px$ for $P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} > 0$, it follows from part (a) that $v(x,t)$ is radially-unbounded positive-definite and decrescent on $\mathbb{R}^2$.
Further, $\dot{v}(x,t) = 2x_1\dot{x}_1 + 4x_2\dot{x}_2 = 2x_1(2x_2 - x_1) + 4x_2(-x_1 - 3x_2)$
$= -2x_1^2 - 12x_2^2 = -x^{\mathrm{T}}\begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}x$ so, by part (a), $-\dot{v}(x,t)$ is positive-definite on $\mathbb{R}^2$. Consequently, by the Lyapunov Global Asymptotic Stability Theorem, the origin is globally asymptotically stable.

[4]

(ii) If we regard (2.3) as $\dot{x} = f(x)$ then $f_x(0) = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ for which the eigenvalues are the solutions of $\det\begin{bmatrix} \lambda & -2 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + 2$. Therefore the eigenvalues are $\pm j\sqrt{2}$ and the Lyapunov Linearization Theorem does not allow us to claim anything about the stability properties of the origin for (2.3) since the eigenvalues are exactly in the imaginary axis.

[2]

The only difference here from part (b-i) concerns $\dot{v}$. It is now
$\dot{v}(x,t) = 2x_1\dot{x}_1 + 4x_2\dot{x}_2 = 2x_1(2x_2 - x_1^3) + 4x_2(-x_1 - 3x_2^5)$
$= -2x_1^4 - 12x_2^6$. So $-\dot{v}(x,t) = 2x_1^4 + 12x_2^6$ and clearly this is strictly positive for all non-zero $x \in G_1$. Therefore $-\dot{v}$ is positive definite on G_1.

[2]

Consequently, by the Lyapunov Asymptotic Stability Theorem, the origin is asymptotically stable.

[1]

(c) {*application of bookwork to a new example*}

Now $s(t) = 0,\ \forall t \ge \tau$, implies that $\dot{e}(t) = -3e(t),\ \forall t \ge \tau$, i.e. that $e(t) = \exp(-3\mathrm{t})e(\tau)$. Hence $e(t) \to 0$ as $t \to \infty$ so the aim $x_1(t) = \exp(-2t)$ is achieved asymptotically.

[3]

Further, $\dot{s}(t) = \ddot{e}(t) + 3\dot{e}(t) = \ddot{x}_1(t) - 4\exp(-2t) + 3(\dot{x}_1(t) + 2\exp(-2t))$
$= -2x_1(t) - 3x_2(t) + u(t) + d_2 - 4\exp(-2t) + 3x_2(t) + 3d_1 + 6\exp(-2t)$
$= -2x_1(t) + u(t) + d_2 + 3d_1 + 2\exp(-2t)$.
Hencethe control u can be chosen so $\dot{s}$ is equal to any desired value ρ. Denote such u by $u_\rho(x(t))$. Usually $s(0)$ will not be zero. If $s(0)$ is positive we apply u_ρ for a negative ρ until s becomes 0. If $s(0) < 0$ we apply u_ρ for a positive ρ until s becomes zero. Once s is zero, we apply u_0 to keep s zero forever, thereby causing the desired behaviour of x_1 to be achieved asymptotically.

[4]

3. (a) (i) {*The heart of this involves a somewhat different approach to a part of a question set last year - and is here to lead into the unseen part (ii)*}

The closed-loop system is

$$\dot{x}(t) = Ax(t) - bb^{\mathrm{T}}Px(t) = [A - bb^{\mathrm{T}}P]x(t)$$

so

$$\begin{aligned}
\dot{v}(t) &= \tfrac{d}{dt}x(t)^{\mathrm{T}}Px(t) \\
&= \dot{x}(t)^{\mathrm{T}}Px(t) + x(t)^{\mathrm{T}}P\dot{x}(t) \\
&= x(t)^{\mathrm{T}}[A - bb^{\mathrm{T}}P]^{\mathrm{T}}Px(t) + x(t)^{\mathrm{T}}P[A - bb^{\mathrm{T}}P]x(t) \\
&= x(t)^{\mathrm{T}}[A^{\mathrm{T}}P+PA]x(t) - 2x(t)^{\mathrm{T}}Pbb^{\mathrm{T}}Px(t) \\
&= x(t)^{\mathrm{T}}[A^{\mathrm{T}}P+PA - 2Pbb^{\mathrm{T}}P]x(t) \qquad (\#) \\
&= x(t)^{\mathrm{T}}[-Q+Pbb^{\mathrm{T}}P - 2Pbb^{\mathrm{T}}P]x(t) \\
&= x(t)^{\mathrm{T}}[-Q-Pbb^{\mathrm{T}}P]x(t) \\
&= -x(t)^{\mathrm{T}}Qx(t) - \|b^{\mathrm{T}}Px(t)\|^2 \\
&\leq -x(t)^{\mathrm{T}}Qx(t).
\end{aligned}$$

Since Q is positive-definite, this shows that $-\dot{v}$ is positive definite. Hence, by the Lyapunov Global Asymptotic Stability Theorem, the origin is globally asymptotically stable. **[5]**

(ii) {*new problem*}

From (#) in part (a-i), but with A replaced by $A+\delta A$, we have

$$\begin{aligned}
\dot{v}(t) &= x(t)^{\mathrm{T}}[(A+\delta A)^{\mathrm{T}}P+P(A+\delta A) - 2Pbb^{\mathrm{T}}P]x(t) \\
&= x(t)^{\mathrm{T}}[A^{\mathrm{T}}P+PA+\{(\delta A)^{\mathrm{T}}P+P\delta A)\} - 2Pbb^{\mathrm{T}}P]x(t) \\
&= x(t)^{\mathrm{T}}[A^{\mathrm{T}}P+PA - 2Pbb^{\mathrm{T}}P]x(t)+x(t)^{\mathrm{T}}\{(\delta A)^{\mathrm{T}}P+P\delta A\}x(t) \\
&= -x(t)^{\mathrm{T}}Qx(t) - \|b^{\mathrm{T}}Px(t)\|^2+x(t)^{\mathrm{T}}\{(\delta A)^{\mathrm{T}}P+P\delta A\}x(t) \\
&\leq -x(t)^{\mathrm{T}}Qx(t)+\|(\delta A)^{\mathrm{T}}P+P\delta A\|\,\|x(t)\|^2 \\
&\leq -x(t)^{\mathrm{T}}Qx(t)+2\|\delta A\|\|P\|\|x(t)\|^2 \\
&\leq \ -\|x(t)\|^2\lambda_{\min}(Q)+2\|\delta A\|\|P\|\|x(t)\|^2 \\
&= \ -\|x(t)\|^2\{\lambda_{\min}(Q)-2\|\delta A\|\|P\|\}
\end{aligned}$$

so $-\dot{v}$ is positive definite on $\mathbb{R}^n$ if $\lambda_{\min}(Q)-2\|\delta A\|\|P\| > 0$,

i.e. if $2\|\delta A\|\|P\| < \lambda_{\min}(Q)$

i.e. if $\|\delta A\| < \frac{1}{2}\lambda_{\min}(Q)/\|P\|$. Hence, by the Lyapunov Global Asymptotic Stability theorem, the origin in globally asymptotically stable for system (3.5) if $\|\delta A\| < \Delta \triangleq \frac{1}{2}\lambda_{\min}(Q)/\|P\|$, **[8]**

(b) {*mostly bookwork with the unseem special case $M=I$ considered at the end*}

Substitution of $A = ZMZ^{\mathrm{T}}$ into $A^{\mathrm{T}}P + PA = -Q$ yields $ZM^{\mathrm{T}}Z^{\mathrm{T}}P + PZMZ^{\mathrm{T}} = -Q$. Pre-multiplying by Z^{T} and post-multiplying by Z gives $Z^{\mathrm{T}}ZM^{\mathrm{T}}Z^{\mathrm{T}}PZ + Z^{\mathrm{T}}PZMZ^{\mathrm{T}}Z = -Z^{\mathrm{T}}QZ$. Since $Z^{\mathrm{T}}Z = I$ owing to the orthogonality of Z, this gives $M^{\mathrm{T}}Z^{\mathrm{T}}PZ + Z^{\mathrm{T}}PZM = -Z^{\mathrm{T}}QZ$.

Let $S = Z^{\mathrm{T}}PZ$. Then S is symmetric and is defined by

$$M^{\mathrm{T}}S + SM = -W \qquad \text{(A3.1)}$$

where $W \triangleq Z^{\mathrm{T}}QZ$. This is generally easier to solve for S than it is to solve $A^{\mathrm{T}}P + PA = -Q$ for P because M is block-upper-triangular with blocks which are either 1×1 or 2×2 and usually A does not have a special structure.

Since $S = Z^{\mathrm{T}}PZ$ and Z is orthogonal, we can find P from S using $P = ZSZ^{\mathrm{T}}$. **[5]**

For the special case $M = D$ with D diagonal and $Q = qI$, (A3.1) gives

$DS + SD = -Z^{\mathrm{T}}qI_nZ = -qI$

which has the solution $S = -\frac{1}{2}qD^{-1}$. So $P = -\frac{1}{2}qZD^{-1}Z^{\mathrm{T}}$. **[2]**

4. (a) {*Modification of bookwork*}

(i) Since $A^2 = 0$ and $Q = I_n$,
$P = Q + A^{\mathrm{T}}QA + (A^{\mathrm{T}})^2QA^2 + (A^{\mathrm{T}})^3QA^3 + \ldots = I + A^{\mathrm{T}}A$.
Hence
$v(x_{k+1}) - v(x_k) = v(Ax_k) - v(x_k) = (Ax_k)^{\mathrm{T}}P(Ax_k) - x_k^{\mathrm{T}}Px_k$
$= x_k^{\mathrm{T}}[A^{\mathrm{T}}PA - P]x_k = x_k^{\mathrm{T}}[A^{\mathrm{T}}(I + A^{\mathrm{T}}A)A - (I + A^{\mathrm{T}}A)]x_k$
$= -x_k^{\mathrm{T}}x_k$. (#)

Hence $v(x_{k+1}) - v(x_k) < 0$ whenever $x_k \neq 0$. **[4]**

Therefore, accepting the given fact that our v is positive-definite etc. on $\mathbb{R}^n$, the discrete-time Lyapanov Global Asymptotic Stability Theorem reveals that the origin is globally asymptotically stable. **[1]**

(ii) Since $v(x_{k+1}) = (Ax_k + bu_k)^{\mathrm{T}}P(Ax_k + bu_k)$ and $P > 0$, choosing u_k to minimize $v(x_{k+1})$ will tend to reduce $\|x_{k+1}\|$ which will tend to increase the rate of convergence of the sequence $\{x_k\}$ to zero. **[1]**

Now

$$v(x_{k+1}) = (Ax_k + bu_k)^{\mathrm{T}}P(Ax_k + bu_k)$$
$$= x_k^{\mathrm{T}}A^TPAx_k + 2x_k^{\mathrm{T}}A^{\mathrm{T}}Pbu_k + u_k^2b^TPb.$$

Since $\partial^2 v(Ax_k+bu_k)/\partial(u_k)^2 = b^{\mathrm{T}}Pb > 0$ (assuming $b \neq 0$) since $P > 0$, choosing u_k so $\partial v(Ax_k+bu_k)/\partial(u_k) = 0$ yields the unconstrained minimizer u_k, which will be denoted here by $\tilde{u}_k$. Hence $\tilde{u}_k = -x_k^{\mathrm{T}}A^{\mathrm{T}}Pb/(b^TPb)$.
Then it is clear that the constrained minimizer $\widehat{u}_k(x_k)$ is $\tilde{u}_k$ if $\tilde{u}_k \in [-1, 2]$, is -1 if $\tilde{u}_k < -1$ and is 2 if $\tilde{u}_k > 2$. **[4]**

Now, for each $k \geq 0$,
$v(Ax_k+b\widehat{u}_k(x_k)) \leq v(Ax_k+b0) = v(Ax_k)$ (£)
since $0 \in [-1, 2]$ but $u_k = 0$ does not necessarily minimize $v(Ax_k+bu_k)$ with respect to $u_k \in [-1, 2]$.
Hence, from (#) and (£) above, for all $k \geq 0$, for the optimally-controlled system:
$v(x_{k+1}) - v(x_k) = v(Ax_k+b\widehat{u}_k(x_k)) - v(x_k) \leq v(Ax_k) - v(x_k)$
$\leq -\|x_k\|^2 < 0$ for all $x_k \neq 0$. Therefore, by the Lyapunov Global Asymptotic Stability Theorem, the origin is globally asymptotically stable for the optimally-controlled system. **[5]**

(b) {*Unseen application of a method for solving the continuous-time Lyapunov equation to solution of the discrete-time Lyapunov equation*}
Now, for a matrix $M \in \mathbb{R}^{2\times 2}$, $\mathrm{vec}(M) = [\,m_{11} \quad m_{12} \quad m_{21} \quad m_{22}\,]^{\mathrm{T}}$ and the Kronecker product of $L \in \mathbb{R}^{2\times 2}$ and $N \in \mathbb{R}^{2\times 2}$ is $L \bigotimes N = \begin{bmatrix} l_{11}N & l_{12}N \\ l_{21}N & l_{22}N \end{bmatrix} \in \mathbb{R}^{4\times 4}$.
Taking vecs, the equation $A^{\mathrm{T}}PA - P = -Q$ becomes $\mathrm{vec}(A^{\mathrm{T}}PA) - \mathrm{vec}(P) = -\mathrm{vec}(Q)$, i.e. $(A^{\mathrm{T}} \bigotimes A^{\mathrm{T}} - I_{n^2})p = -q$ where $p = \mathrm{vec}(p)$ and $q = \mathrm{vec}(Q)$.
Since the solution P of $A^{\mathrm{T}}PA - P = -Q$ is unique, the solution p of $(A^{\mathrm{T}} \bigotimes A^{\mathrm{T}} - I_{n^2})p = -q$ is also unique. Hence, since $A^{\mathrm{T}} \bigotimes A^{\mathrm{T}} - I_{n^2}$ is square, the null-space of $A^{\mathrm{T}} \bigotimes A^{\mathrm{T}} - I_{n^2}$ must equal just $\{0\}$ so $A^{\mathrm{T}} \bigotimes A^{\mathrm{T}} - I_{n^2}$ is non-singular.
Therefore, conceptually at least, there is no problem in solving $(A^{\mathrm{T}} \bigotimes A^{\mathrm{T}} - I_{n^2})p = -q$ for p and the required solution P of $A^{\mathrm{T}}PA - P = -Q$ is then $\mathrm{vec}^{-1}(p)$. **[5]**

5. (a) (i) {*Apart from the initial definition, this is a bit new for them*}

A subsystem with initial condition x_o, input e and output u is strictly input passive if there is a scalar $\beta(x_o) \le 0$ and a scalar $\delta > 0$ such that $\int_0^\tau y(t)u(t)dt$ $\ge \beta(x_o) + \delta\int_0^\tau u(t)^2 dt$, $\forall \tau \ge 0, \forall u \in \mathcal{L}_{2e}$.

Now consider H when $y(t) = \phi(u(t), t)$ for all t. For $\phi \in$ sector$[\alpha, \beta]$:

$\alpha u^2 \le \phi(u,t)u \le \beta u^2$.

Hence $\int_0^\tau y(t)u(t)dt = \int_0^\tau \phi(u(t),t)u(t)dt \ge \int_0^\tau \alpha u(t)^2 dt = \delta\int_0^\tau u(t)^2 dt + \beta(x_o)$

where $\delta = \alpha$ and $\beta(x_o) = 0$.

Hence H is strictly input passive.

Similarly, since $\phi \in$ sector$[\alpha, \beta]$, we have $\alpha \le \frac{\phi(u(t),t)}{u(t)} \le \beta$ whenever $u(t) \neq 0$ so $|\phi(u,t)| \le \beta\,|u|$. Consequently $\int_0^T y(t)^2 dt = \int_0^T \phi(u(t),t)^2 dt \le \int_0^T \beta^2 u(t)^2 dt$

so $\|y_T\|_{\mathcal{L}_2} \le \beta\|u_T\|_{\mathcal{L}_2}$. **[5]**

(ii) {*apart from the initial definition, this is a bit new but since it is quite complicated they will not find it straightforward unless they know well what they are doing*}

H is strictly output passive if there is a scalar $\beta(x_o) \le 0$ and a scalar $\delta > 0$ such that $\int_0^\tau y(t)u(t)dt \ge \beta(x_o) + \delta\int_0^\tau y(t)^2 dt$, $\forall\tau \ge 0, \forall u \in \mathcal{L}_{2e}$.

For our particular H:

$\int_0^\tau \frac{d}{dt}x(t)^{\mathrm{T}}Px(t)dt = x(t)^{\mathrm{T}}Px(t)|_0^\tau = x(\tau)^{\mathrm{T}}Px(\tau) - x_o^{\mathrm{T}}Px_o \ge -x_o^{\mathrm{T}}Px_o$ (\$)

since $x(\tau)^{\mathrm{T}}Px(\tau) \ge 0$ owing to the fact that P is positive-definite.

Also:

$$\int_0^\tau \frac{d}{dt}x(t)^{\mathrm{T}}Px(t)dt = \int_0^\tau \{\dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x}\}dt$$
$$= \int_0^\tau \{[Ax+bu]^{\mathrm{T}}Px + x^{\mathrm{T}}P[Ax+bu]\}dt$$
$$= \int_0^\tau \{x^{\mathrm{T}}[A^{\mathrm{T}}P+PA]x + 2ub^{\mathrm{T}}Px\}dt = \int_0^\tau \{-x^{\mathrm{T}}[cc^{\mathrm{T}}]x + 2ub^{\mathrm{T}}Px\}dt$$
$$= \int_0^\tau \{-(c^{\mathrm{T}}x)^2 + 2uc^{\mathrm{T}}x\}dt = \int_0^\tau \{-y^2 + 2uy\}dt. \qquad (£)$$

From (\$) and (£),

$\int_0^\tau uy\,dt \ge \frac{1}{2}\int_0^\tau y^2 dt - \frac{1}{2}x_o^{\mathrm{T}}Px_o = \delta_2\int_0^\tau y^2 dt + \beta_2(x_o)$

where $\delta_2 = \frac{1}{2} > 0$ and $\beta_2(x_o) = -\frac{1}{2}x_o^{\mathrm{T}}Px_o \le 0$.

Hence H is strictly output passive. **[7]**

(b) {*repackaged tutorial question which they will not recognise*}

Now

$$\|u_T\|_{\mathcal{L}_2}\|r_T\|_{\mathcal{L}_2} \ge \text{(by Cauchy-Schwartz)} \int_0^T u(t)r(t)dt$$
$$= \int_0^T u(t)\{e(t)+y(t)\}dt = \int_0^T u(t)e(t)dt + \int_0^T u(t)y(t)dt. \qquad (¥)$$

Here $\int_0^T u(t)e(t)dt \ge \beta_1$ for $\beta_1 \le 0$ since H_1 is passive, and $\int_0^T u(t)y(t)dt \ge \beta_2(x_o) + \delta_2\int_0^T y(t)^2 dt$ where $\beta_2 \le 0$ and $\delta_2 > 0$ since H_2 is strictly output passive.

Use of these in (¥) yields:

$\|u_T\|_{\mathcal{L}_2}\|r_T\|_{\mathcal{L}_2} \ge \int_0^T u(t)e(t)dt + \int_0^T u(t)y(t)dt \ge \beta_1 + \beta_2(x_o) + \delta_2\int_0^T y(t)^2 dt$.

For our case with $r \equiv 0$, this gives

$0 \ge \beta + \delta_2\int_0^T y(t)^2 dt$ where $\beta = \beta_1 + \beta_2(x_o) \le 0$ and $\delta_2 > 0$, i.e. $\|y_T\|_{\mathcal{L}_2}^2 \le -\frac{\beta}{\delta} < \infty$, $\forall T < \infty$, as required. **[8]**

6. (a) *{this is an application of material they know for the case* $yu = f\dot{f}$ *to a more complicated case that will help with the solution of part (b) below}*

The system is passive if there is a $\beta \le 0$ such that

$\int_0^\tau y(t)u(t)dt \ge \beta$ for all $\tau \ge 0$ and for all input functions u.

If $y(t)u(t)$ can be written as $\alpha f(t)\dot{f}(t)$ with $\alpha > 0$, then

$\int_0^\tau y(t)u(t)dt = \alpha\int_0^\tau f(t)\dot{f}(t)dt = \frac{1}{2}\alpha\int_0^\tau \frac{d}{dt}\{f(t)^2\}dt = \frac{1}{2}\alpha\{f(\tau)^2 - f(0)^2\}$

$\ge -\frac{1}{2}\alpha f(0)^2 \triangleq \beta$ with $\beta \le 0$, so the system is passive.

If $y(t)u(t) = (1-\alpha\gamma(t))p(t)$ then this can be written as $y(t)u(t) = \alpha\frac{(\alpha^{-1}-\gamma(t))}{g}gp(t)$ and this can be written as $\alpha f(t)\dot{f}(t)$ with $f(t) = \frac{(\alpha^{-1}-\gamma(t))}{g}$ and $\dot{f}(t) = gp(t)$. Then we obtain passivity if $-\frac{\dot{\gamma}}{g} = gp(t)$, i.e. if $\dot{\gamma}(t) = -g^2p(t)$. **[5]**

(b) *{new case - probably they are expecting a question of this type however it is still a searching question that cannot be done without understanding}*

The Controlled plant is $\dot{x} = Ax + \alpha b\gamma(t)\{r(t) - f^{\mathrm{T}}x\}$.

Perfect model following is possible since the choice $\gamma(t) = \alpha^{-1}$ causes the equation of the controlled plant to become $\dot{x} = Ax + b\{r(t) - f^{\mathrm{T}}x\}$ which is the equation for the reference model. **[2]**

The error subsystem: Define $e = \overline{x} - x$. Then

$$\begin{aligned} \dot{e} = \dot{\overline{x}} - \dot{x} &= \overline{A}\overline{x} + br - \{Ax + \alpha b\gamma(r - f^{\mathrm{T}}x)\} \\ &= \overline{A}(\overline{x} - x) + \overline{A}x + br - \{Ax + \alpha b\gamma(r - f^{\mathrm{T}}x)\} \\ &= \overline{A}e + Iw \end{aligned}$$

where

$$\begin{aligned} w &= \overline{A}x + br - \{Ax + \alpha b\gamma(r(t) - f^{\mathrm{T}}x)\} \\ &= -bf^{\mathrm{T}}x + br - \alpha\gamma br(t) + \alpha\gamma bf^{\mathrm{T}}x \\ &= (\alpha\gamma - 1)b(f^{\mathrm{T}}x - r). \end{aligned}$$

x Solve $\overline{A}^{\mathrm{T}}P + P\overline{A} = -I$ for P, giving a positive definite P since $\overline{A}$ is a stability matrix. Choose the output matrix to be $b^{\mathrm{T}}P$. Then, from the Kalman-Popov-Yakubovic Lemma, the error subsystem of Figure A6.1 is strictly output passive. **[6]**

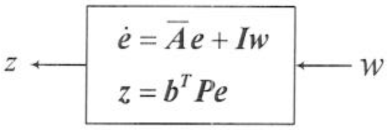

Figure A6.1

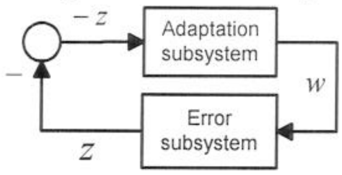

Figure A6.2

Design of adaptation law

We view the situation as in Figure A6.2 and require the connection between $-z$ and w to be passive so we consider

$$\begin{aligned} \int_0^\tau (-z(t)^{\mathrm{T}}w(t)dt &= \int_0^\tau -z(t)^{\mathrm{T}}\{\alpha\gamma(t) - 1\}b\{f^{\mathrm{T}}x(t) - r(t)\}dt \\ &= \int_0^\tau \{1 - \alpha\gamma(t)\}z(t)^{\mathrm{T}}b\{f^{\mathrm{T}}x(t) - r(t)\}dt \\ &= \int_0^\tau \{1 - \alpha\gamma(t)\}p(t)dt \end{aligned}$$

where

$$p(t) = z(t)^{\mathrm{T}}b\{f^{\mathrm{T}}x(t) - r(t)\}.$$

From part (a), this is passive if

$$\dot{\gamma}(t) = -g^2p(t)$$

where $g > 0$.

So we just need to use the adaptation law

$\gamma(t) = -g^2\int_0^t z(t)^{\mathrm{T}}b\{f^{\mathrm{T}}x(t) - r(t)\}$.

Then, by a Passivity Theorem, $\|\overline{x} - x\|_{\mathcal{L}_2} < \infty$, as required. **[7]**