IMPERIAL COLLEGE LONDON

E4.23
C1.4
ISE4.27

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

MSc and EEE/ISE PART IV: MEng and ACGI

STABILITY AND CONTROL OF NON-LINEAR SYSTEMS

Monday, 12 May 10:00 am

Time allowed: 3:00 hours

Corrected Copy

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : J.C. Allwright
Second Marker(s) : E.C. Kerrigan

Information for invigilators: none

Information for candidates:

$\|x\|$ denotes $\sqrt{x^{\mathrm{T}}x}$ for $x \in \mathbb{R}^n$

$\|x\|_{\mathcal{L}_2}$ denotes $\sqrt{\int_0^\infty x(t)^{\mathrm{T}}x(t)dt}$ for $x \in \mathcal{L}_2$

x_τ denotes the truncation of x onto $[0, \tau]$

$\mathcal{L}_{2e}$ denotes the extension of $\mathcal{L}_2$

Unless clear from the context: $\varepsilon(t)$, $y(t)$, $u(t)$ and $r(t)$ are all scalar valued

For a symmetric matrix P: $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of P, respectively

* denotes complex conjugate

$\triangleq$ means 'is defined equal to'

I_n is the $n \times n$ identity matrix.

The Questions

1. (a) Here $x \in \mathbb{R}$.

(i) Consider the systems

$$\ddot{x} = f_1(x, \dot{x}) \triangleq -x + 1; \qquad \ddot{x} = f_2(x, \dot{x}) \triangleq -x.$$

By adapting the general form of solution for such systems, and without using isoclines, sketch typical trajectories for these systems in the $(x, \dot{x})$ phase-space. [3]

(ii) Consider also the systems

$$\ddot{x} = f_3(x, \dot{x}) \triangleq -\dot{x}; \qquad \ddot{x} = f_4(x, \dot{x}) \triangleq \dot{x}.$$

Determine the corresponding phase-plane differential equations and use them to sketch typical trajectories for each system in the $(x, \dot{x})$ phase-plane. [3]

(iii) Use the results of parts (a-i) and (a-ii) above to sketch the complete closed trajectory (starting and returning later to the starting point) that has the initial condition $x(0) = 0, \dot{x}(0) = 3$ for the system

$$\ddot{x} = \begin{cases} f_1(x, \dot{x}) & \text{if } x > 1 \\ f_2(x, \dot{x}) & \text{if } x < 0. \\ f_3(x, \dot{x}) & \text{if } x \in [0,1] \text{ and } \dot{x} \geq 0 \\ f_4(x, \dot{x}) & \text{if } x \in [0,1] \text{ and } \dot{x} < 0. \end{cases}$$

Here the f_i are those of parts (a-i) and (a-ii) above. [4]

What is the value of t at which x becomes equal to 1 for the second time?.

Hint: $\int \frac{1}{\gamma x + \delta} dx = \frac{1}{\gamma} \ln(\gamma x + \delta)$. [3]

(b) In Figures 1.1 and 1.2 below: n is a skew-symmetric function.

(i) Consider Figure 1.1. Suppose $e(t) = a\sin(\omega t)$ for scalar $a \geq 0$ and all $t \geq 0$. Outline the way in which a Fourier series representation of u over one period of $\sin(\omega t)$ can be used to obtain an approximation to u that has the form $\overline{a}\sin(\omega t + \phi)$ for values of $\overline{a}$ and ϕ which should be specified. Hence derive briefly a general formula for the describing-function $N(a)$ of n. [4]

(ii) Consider Figure 1.2. State the harmonic balance equation. Outline a graphical method for predicting whether e will oscillate. How can you predict the amplitude and frequency of such an oscillation? [3]

Figure 1.1

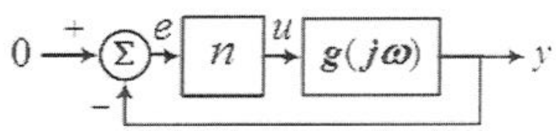

Figure 1.2

2. (a) Consider the function $v(x,t) = x^{\mathrm{T}}Px$ where $x \in \mathbb{R}^n$ and $P^{\mathrm{T}} = P > 0$.
Show that v is radially-unbounded positive-definite and decrescent on $\mathbb{R}^n$. **[4]**

(b) (i) Consider the system

$$\begin{aligned} \dot{x}_1 &= 2x_2 - x_1 \\ \dot{x}_2 &= -x_1 - 3x_2 \end{aligned} \qquad (2.1)$$

and the function

$$v(x,t) = x_1^2 + 2x_2^2. \qquad (2.2)$$

Why is the origin an equilibrium state for this system?
Write the strongest stability result for the origin that can be obtained using v, the result of part (a) and Lyapunov theory. Justify your application of the relevant theorem. **[4]**

(ii) Consider v of (2.2) and the following modified version of system (2.1):

$$\begin{aligned} \dot{x}_1 &= 2x_2 - x_1^3 \\ \dot{x}_2 &= -x_1 - 3x_2^5. \end{aligned} \qquad (2.3)$$

What can be said about the stability properties of the origin for (2.3) using the Lyapunov Linearization Theorem? Justify your answer. **[2]**

For (2.3), show that $-\dot{v}$ is positive definite on the set

$$G_1 = \{x \in \mathbb{R}^2 : \|x\|_2 \le 1\}. \qquad \textbf{[2]}$$

Obtain the strongest stability result for the origin regarding system (2.3) that can be proved using part (a) and the above property of $-\dot{v}$. **[1]**

(c) This part concerns the application of sliding mode control to the system

$$\dot{x}(t) = Ax(t) + bu(t) + d : x(0) = x_0$$

where $d, x \in \mathbb{R}^2$ and $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with constant d.
Suppose it is desired that $x_1(t) = \exp(-2t)$ when $t \ge 0$.
Let

$$e(t) = x_1(t) - \exp(-2t)$$

and

$$s(t) = \dot{e}(t) + 3e(t).$$

What useful outcome is obtained by arranging that $s(t) = 0$ for all t greater than a particular time τ? Justify your answer. **[3]**
Show that there are control laws that will reduce s to zero and keep it at zero. **[4]**

3. Here $A \in \mathbb{R}^{n\times n}$ is a stability matrix if the real part of each eigenvalue of A is strictly negative.

(a) Consider the model of a system given by

$$\dot{x}(t) = Ax(t) + bu(t) : x(0) = x_o. \qquad (3.1)$$

Here A is a stability matrix and $b,\ x \in \mathbb{R}^n$. The control u is scalar-valued.

(i) Suppose $P \in \mathbb{R}^{n\times n}$ satisfies the equation

$$A^{\mathrm{T}}P + PA + Q - Pbb^{\mathrm{T}}P = 0 \qquad (3.2)$$

where $P = P^{\mathrm{T}} > 0$ and $Q \in \mathbb{R}^{n\times n}$ with $Q = Q^{\mathrm{T}} > 0$.

Use the radially-unbounded positive-definite decrescent function

$$v(x,t) = x^{\mathrm{T}}Px \qquad (3.3)$$

to show that, when the control law

$$u(t) = -b^{\mathrm{T}}Px(t) \qquad (3.4)$$

is applied, the origin is globally asymptotically stable for system (3.1). **[5]**

(ii) Now suppose that, owing to a modelling error, the system is actually

$$\dot{x}(t) = (A+\delta A)x(t) + bu(t) : x(0) = x_o \qquad (3.5)$$

where $\delta \mathrm{A} \in \mathbb{R}^{n\times n}$.

Use the P. v and u defined in part (a-i) to show that the origin is globally asymptotically stable for the actual system of (3.5) if $\|\delta A\| < \Delta$ for an appropriate strictly positive Δ, which should be specified. **[8]**

(b) Suppose the matrix $A \in \mathbb{R}^{n\times n}$ has the real Schur factorization $A = ZMZ^{\mathrm{T}}$ where Z is orthogonal.

Apply the method used by Bartels and Stuart to exploit this factorization in order to transform the matrix Lyapunov equation

$$A^{\mathrm{T}}P + PA = -Q \qquad (3.6)$$

into an equation which is more easy to solve. Here $A,\ P,\ Q \in \mathbb{R}^{n\times n}$ with $P^{\mathrm{T}} = P > 0$ and $Q^{\mathrm{T}} = Q > 0$.

State the way in which you would determine the solution P of (3.6) from the solution of the transformed equation. **[5]**

Use this method to determine P when M is a non-singular diagonal stability matrix and $Q = qI_n$ with q a strictly positive scalar and I_n the identity matrix from $R^{n\times n}$. **[2]**

4. Consider the system

$$x_{k+1} = Ax_k + bu_k : \quad x_0 = x_o$$

where $b,\ x_k \in \mathbb{R}^n$ and where $A \in \mathbb{R}^{n\times n}$ is a discrete-time stability matrix in that the modulus of each eigenvalue of A is strictly less than 1.

For symmetric positive-definite $Q \in \mathbb{R}^{n\times n}$, let

$$P = Q + A^{\mathrm{T}}QA + (A^{\mathrm{T}})^2QA^2 + (A^{\mathrm{T}})^3QA^3 + \dots$$

and

$$v(x) = x^{\mathrm{T}}Px.$$

(a) Suppose $n > 2$ and $Q = I_n$ and $A^2 = 0$, where I_n is the $n \times n$ identity matrix. You may assume without proof that v is radially-unbounded positive-definite and decrescent on $\mathbb{R}^n$.

(i) Determine the value of P. Show that if the u_k are all zero then

$$v(x_{k+1}) - v(x_k) \leq -\|x_k\|^2,\ \forall k \geq 0. \qquad [4]$$

Hence prove global asymptotic stability of the origin using a Lyapunov theorem. [1]

(ii) Now suppose that each u_k is chosen to be $\widehat{u}_k(x_k)$ where this minimizes $v(Ax_k + bu_k)$ with respect to $u_k \in [-1, 2]$.

Why might one wish to use such an optimal control? [1]

Derive a simple method for finding $\widehat{u}_k(x_k)$ that takes advantage of the simple form of v. [4]

Modify the analysis of part (a-i) to show that the origin is globally asymptotically stable for the optimally controlled system. [5]

(b) Suppose $n = 2$ and the matrix Lyapunov equation

$$A^{\mathrm{T}}PA - P = -Q$$

has a unique symmetric solution P.

Define the vec of a matrix $M \in \mathbb{R}^{2\times 2}$ and the Kronecker product of two matrices from $\mathbb{R}^{2\times 2}$.

Use vecs and a Kronecker product to determine P in terms of the solution of a linear equation $Mf = g$ for appropriate M, f and g, which should be specified.

Why is your M non-singular? [5]

5. (a) Consider the system H of Figure 5.1 where u and y are scalar valued.

(i) Define strict input passivity for H.

Suppose the output y of H is given by $y(t) = \phi(u(t),t)$ where

$\phi \in$ sector[α,β] with $0 < \alpha < \beta < \infty$.

Show that H is strictly input passive and that

$$\|y_T\|_{\mathcal{L}_2} \le \beta \|u_T\|_{\mathcal{L}_2}, \forall T \in [0,\infty), \forall u \in \mathcal{L}_{2e}.$$ **[5]**

(ii) Define strict output passivity for H.

Suppose H consists of a time-invariant system

$$\dot{x} = Ax + bu : x(0) = x_0$$
$$y = c^{\mathrm{T}} x$$

where the real part of each eigenvalue of A is strictly negative.

Suppose $P \in \mathbb{R}^{n \times n}$ is symmetric and positive definite and solves

$$A^T P + PA = -cc^{\mathrm{T}}$$

and suppose that

$$c = Pb.$$

By considering $\int_0^T \frac{d}{dt} x(t)^{\mathrm{T}} P x(t) dt$, show that H is strictly output passive. **[7]**

(b) Consider the system of Figure 5.2 below, where H_1 is passive and H_2 is strictly output passive and the system equations have a unique solution whenever $r \in \mathcal{L}_{2e}$.

The subsystem H_1 does not contain any dynamics and x_o represents the initial condition for H_2.

Suppose $r(t) = 0$ for all t.

Show that $\|y_T\|_{\mathcal{L}_2} < \infty$ for all finite T.

Hint: start by considering $\int_0^T u(t)r(t)dt$ for general r. **[8]**

Figure 5.1

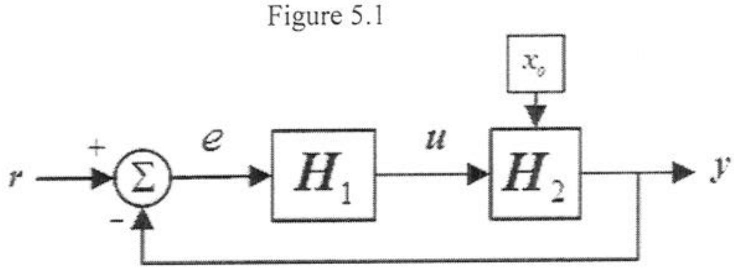

Figure 5.2

6. (a) Consider a SISO system with input $u(t)$ and output $y(t)$.

Show that the system is passive if $y(t)u(t) = \alpha f(t)\dot{f}(t)$ for scalar $\alpha > 0$.

Now suppose $y(t)u(t) = (1-\alpha\gamma(t))p(t)$ for scalar α, $\gamma(t)$, $p(t)$ with $\alpha > 0$.

Write $(1-\alpha\gamma(t))p(t)$ as $\alpha f(t)\dot{f}(t)$ with $f(t) = \frac{\alpha^{-1}-\gamma(t)}{g}$ for scalar $g > 0$ and use the above to show that the system is passive if γ is chosen so that $\dot{\gamma}(t) = -g^2 p(t)$. **[5]**

(b) Consider a plant modelled by

$$\dot{x}(t) = Ax(t) + \alpha bu(t)$$

where α is a strictly positive scalar with unknown value.

Suppose you would like the plant to behave like the reference model

$$\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + br(t)$$

where $\overline{A} \triangleq A - bf^{\mathrm{T}}$ and f has been chosen so that each eigenvalue of $\overline{A}$ has real part that is strictly negative.

The control law

$$u(t) = \gamma(t)[r(t) - f^{\mathrm{T}}x(t)]$$

is to be applied to the plant with γ chosen adaptively.

Show that perfect model following is possible by suitable choice of γ. **[2]**

Let

$$e = \overline{x} - x.$$

Show that

$$\dot{e}(t) = \overline{A}e(t) + w(t)$$

where

$$w(t) = \{\alpha\gamma(t) - 1\}b\{f^{\mathrm{T}}x(t) - r(t)\}.$$

Determine an error subsystem, with a suitable output z, that is strictly output passive.

There is no need to solve any Lyapunov equations that are involved. **[6]**

Then, possibly using a result from part (a) above, derive an adaptation law for $\gamma(t)$ such that $\|\overline{x} - x\|_{\mathcal{L}_2} < \infty$.

Give sufficient detail to make your method clear. **[7]**