IMPERIAL COLLEGE LONDON

E4.23
CS2.1
ISE4.27

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

STABILITY AND CONTROL OF NON-LINEAR SYSTEMS

Thursday, 14 May 10:00 am

Time allowed: 3:00 hours

One correction made

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D. Angeli

Second Marker(s) : E.C. Kerrigan

STABILITY AND CONTROL OF NONLINEAR SYSTEMS

1. Consider the second order nonlinear differential equation:

$$\ddot{y}(t) = \mathrm{atan}(y(t)) - \frac{\dot{y}(t)}{1+y^2(t)} - \frac{\pi}{4}y(t),$$

defined for all $y \in \mathbb{R}$.

a) Choose a suitable state variable and write the corresponding state-space model. [4]

b) Compute all equilibria of the system. [4]

c) Linearize the system around each of the equilibria determined in part b) and classify the corresponding local phase-plane portrait (SADDLE, NODE, FOCUS, CENTER, STABLE, UNSTABLE). [6]

d) Exploiting the local information obtained in part c), sketch a consistent global phase portrait for the system. [6]

2. Consider the three dimensional nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^3 - \tfrac{x_2}{x_1^2+1} - x_3 - x_2 - x_1, \\ \dot{x}_3 &= x_1^3 + \tfrac{x_2}{1+x_1^2} + x_3 + x_1. \end{aligned}$$

a) Show that $y = x_1 + x_2 + x_3$ is constant along solutions. [4]

b) Write the equations of the bidimensional system obtained for $x_1 + x_2 + x_3 = 0$. (Hint: use the coordinates x_1 and x_2) [4]

c) Compute the unique equilibrium of the system determined in part b) and show, using a candidate Lyapunov function $V(x_1, x_2) = \alpha x_1^a + \beta x_2^b$, that this equilibrium is Globally Asymptotically Stable (choose the real parameters α, β and the integers a, b in a suitable way). [6]

d) Can local stability properties of the system determined in part b) be assessed by Lyapunov's linearization method? Explain your answer. [6]

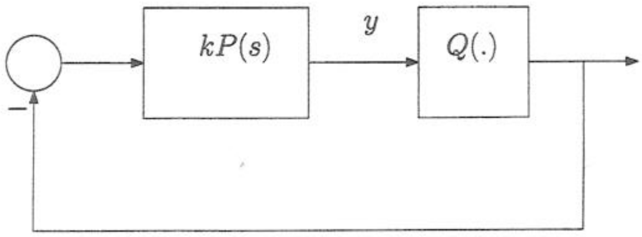

Figure 3.1 Closed loop system

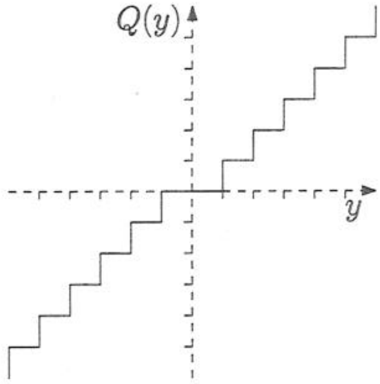

Figure 3.2 Quantization device: Input-Output map (assume equal units on both axis)

3. A SISO linear plant with transfer function $P(s)$ is controlled by means of a proportional controller k. Let $P(s) = \frac{1}{s^2+s+1}$. Due to the presence of a nonlinear static quantization device on the sensor, the overall feedback loop is as in Figure 3.1, where $Q(\cdot)$ is the nonlinear element with the characteristic given in Figure 3.2.

   a) What is the smallest sector which comprises the quantization nonlinearity (assuming the same units on the two axis)? [4]

   b) Draw the Nyquist plot of $P(s)$ and find out what is the maximum value of k which does not destabilize the system in the absence of quantization. [8]

   c) What is the maximum value of k allowed by the circle criterion in order to preserve stability in the presence of quantization? [8]

4. Consider the time-invariant linear system:

$$\dot{x} = Ax$$

with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ together with the following Statement:

Statement: If A is diagonalizable, there exists $P > 0$ so that
$\frac{d}{dt} x' P x \leq 2\lambda_{\max} x' P x$
with $\lambda_{\max} = \max\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{sp}(A)\}$ and $\mathrm{sp}(A)$ denoting the spectrum of the matrix A.

a) Show that the Statement is true. Hint: build first P for the simple systems:

$$\dot{x} = \lambda x \qquad x \in \mathbb{R},$$

$$\dot{x} = \begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix} x \qquad x \in \mathbb{R}^2.$$

[8]

b) Show, by means of an example, that if A is not diagonalizable, there is no $P > 0$ such that an inequality as in the Statement holds. [5]

c) Argue that $V(x) = x' P x$ as given in the Statement can be used to prove global exponential stability, provided A is Hurwitz. [7]

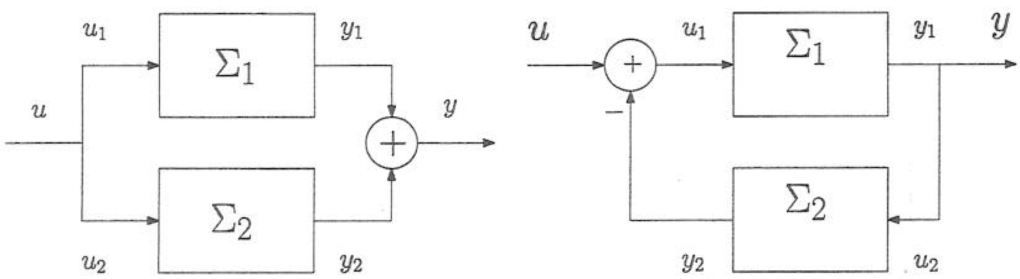

Figure 5.1 Interconnected systems

5. Consider a nonlinear time-invariant system

$$\dot{x} = f(x,u), \qquad y = h(x).$$

a) Recall the time-domain definition of passivity of a nonlinear system. [2]

b) Consider the interconnected systems shown in the Figure 5.1. Show that each one of them is again a passive system, provided the individual subsystems are such. [2]

c) Show, by means of an example, that the series of passive systems need not be passive. [4]

d) Can you think of one input-output pair which violates the definition of passivity for a series of passive linear systems? [4]

e) Consider the following nonlinear circuital components:

i) Nonlinear resistor with characteristic equation:

$$V = R(I)$$

ii) Nonlinear inductor with characteristic equation:

$$L(I)\dot{I} = V$$

iii) Nonlinear capacitor with characteristic equation:

$$C(V)\dot{V} = I$$

Find conditions on the nonlinear functions $R(\cdot)$, $L(\cdot)$ and $C(\cdot)$ so that the resulting components are passive with respect to V and I as input and output variables. [4]

f) Show that the network obtained by composing in series an inductor and a capacitor (in the sense of circuit theory) is lossless. [4]

6. Consider the parameter-dependent nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -k\sin(x_1) + x_2, \\ \dot{x}_2 &= \text{atn}(x_2) + x_3, \\ \dot{x}_3 &= u,\end{aligned}$$

atan (x_2) + x_3,

with state $x = [x_1, x_2, x_3]$ taking values in $\mathbb{R}^3$ and control u taking values in $\mathbb{R}$.

a) Show that the system with output $y = x_1$ has relative degree 3. [3]

b) Is the system globally feedback linearizable? Why? [2]

c) Build a global feedback stabilizer assuming k is known. [3]

d) Let $y = x_2$. Compute the relative degree and discuss if the system can be globally stabilized by means of input-output feedback linearization. What are the zero-dynamics? Are they Input-to-State Stable? Design a local feedback stabilizer by means of Input-Output feedback linearization. How many equilibria has the closed-loop system? [6]

e) Assume now k is only known to belong to the interval $[-\varepsilon, +\varepsilon]$. Design by means of backstepping a controller which robustly globally asymptotically stabilizes the origin irrespectively of the value of k. (Hint: find a robust virtual control for the x_1 equation.) [6]

E4.23 / ISE 4.27 / CS2.1

STABILITY AND CONTROL OF NONLINEAR SYSTEMS
MODEL ANSWERS 2009

1. Exercise

a) We may choose the following state variable $x(t) = [y(t), \dot{y}(t)]' \doteq [x_1(t), x_2(t)]'$. With such choice the model of the system reads:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \operatorname{atan}(x_1) - \tfrac{x_2}{1+x_1^2} - \tfrac{\pi}{4}x_1. \end{aligned}$$

b) The equilibria are obtained solving:

$$\begin{cases} x_2 = 0 \\ \operatorname{atn}(x_1) - \tfrac{x_2}{1+x_1^2} - \tfrac{\pi}{4}x_1 = 0 \end{cases}$$

Substituting $x_2 = 0$ in the second equation we get:

$$\operatorname{atan}(x_1) = \frac{\pi}{4}x_1$$

that is $x_1 = -1, 0, 1$. We have therefore 3 possible equilibria: $[-1,0]', [0,0]', [1,0]'$.

c) To compute the linearization $\dot{\delta x} = \left.\frac{\partial f}{\partial x}\right|_{x=x_e} \delta x$ around such points note that:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{1}{1+x_1^2} - \frac{\pi}{4} + \frac{2x_1x_2}{(1+x_1^2)^2} & -\frac{1}{1+x_1^2} \end{bmatrix}.$$

Evaluating the above expression at equilibria, yields

$$\left.\frac{\partial f}{\partial x}\right|_{x=[\pm 1,0]'} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} - \frac{\pi}{4} & -\frac{1}{2} \end{bmatrix}$$

and

$$\left.\frac{\partial f}{\partial x}\right|_{x=[0,0]'} = \begin{bmatrix} 0 & 1 \\ 1 - \frac{\pi}{4} & -1 \end{bmatrix}.$$

Computing the characteristic polynomial of the first matrix yields:

$$\chi(s) = s^2 + \frac{s}{2} + \frac{\pi - 2}{4}$$

which admits two roots with negative real part; moreover the discriminant is given by: $\frac{1}{4} - (\pi - 2) = 2.25 - \pi < 0$. This means the equilibria are stable foci. For $x_e = [0,0]'$ we have:

$$\chi(s) = s^2 + s - \frac{4 - \pi}{4}.$$

Hence, solutions have respectively positive and negative real parts, and are real. This is therefore a saddle point.

d) Without further analysis we may conjecture a phase plot along the lines of the Figure 1.1.

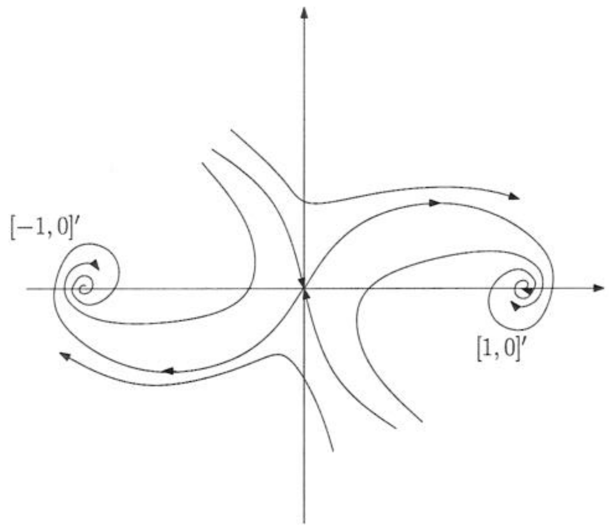

Figure 1.1 Qualitative phase portrait

2. Exercise

a) In order to show that $y(t)$ is constant for all initial conditions, it is enough to show $\dot{y} = 0$. Hence we compute: $\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$.

b) Next we substitute $x_3 = -x_1 - x_2$ in the systems equations yielding:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & -x_1^3 - \frac{x_2}{1+x_1^2}. \end{array}$$

c) The equilibrium is obtained solving:

$$\left\{ \begin{array}{rcl} x_2 & = & 0 \\ -x_1^3 - \frac{x_2}{1+x_1^2} & = & 0 \end{array} \right.$$

Substituting $x_2 = 0$ in the second equation yields, $x_1^3 = 0$, that is $x_1 = 0$. Hence, there exists a unique equilibrium in $[0,0]'$. Let us verify that $V(x_1,x_2)$ is a suitable candidate Lyapunov function to prove global asymptotic stability. We take $V(x_1,x_2) = x_1^4/4 + x_2^2/2$. Indeed V is differentiable, and positive definite:

$$x \neq 0 \Rightarrow x_1 \neq 0 \text{ or } x_2 \neq 0 \Rightarrow \left\{ \begin{array}{l} \text{in the first case } V(x_1,x_2) \geq \frac{x_1^4}{4} > 0 \\ \text{otherwise } V(x_1,x_2) \geq \frac{x_2^2}{2} > 0 \end{array} \right.$$

So V is positive definite. It is straightforward to verify that V is radially unbounded. Next we compute $\dot{V}$.

$$\dot{V} = \dot{x}_1 x_1^3 + \dot{x}_2 x_2 = -\frac{x_2^2}{1+x_1^2} \leq 0$$

Hence, solutions are bounded and by Lasalle's principle converge to the largest invariant set contained in $K \doteq \{(x_1,x_2)' : x_2 = 0\}$. We claim that the only invariant set contained in K is actually the equilibrium itself. Indeed, asking for $\dot{x}_2 = 0$ simultaneously to $x_2 = 0$ yields $x_1 = 0$.

d) We now proceed to linearizing the system. The Jacobian is given by:

$$\frac{\partial f}{\partial x} = \left[\begin{array}{cc} 0 & 1 \\ -3x_1^2 + \frac{2x_1x_2}{(1+x_1^2)^2} & \frac{2x_2}{1+x_1^2} \end{array} \right]$$

which evaluated at $[0,0]'$ yields

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0,x_2=0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We do have a double eigenvalue at 0, that is on the imaginary axis. This is, henceforth, a critical case in which we cannot appeal to the Lyapunov theorem to claim local asymptotic stability of the origin.

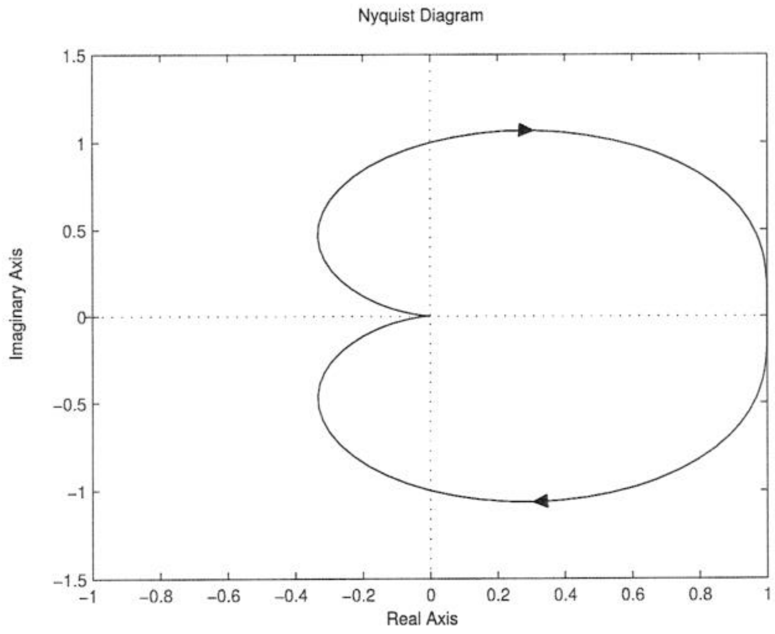

Figure 3.1 Nyquist plot of $P(s)$

3. Exercise

a) The smallest sector containing the quantization nonlinearity is $[0,1]$ (notice the local slope at the origin is 0 !).

b) The Nyquist plot of $P(s)$ is as in Figure 3.1. Hence, for all $k > 0$ the resulting closed-loop system is asymptotically stable (the point $-1/k$ is never encircled by the diagram).

c) Due to the effect of quantization, however, and applying circle criterion, we are only guaranteed GAS, for all ks such that the vertical line through $-\frac{1}{k}$ does not meet the Nyquist diagram. Hence we need to compute the minimum value of the real part of $P(j\omega)$ as $\omega \in \mathbb{R}$.

$$\mathrm{Re}[P(j\omega)] = \frac{1-\omega^2}{(1-\omega^2)^2+\omega^2}$$

Zeroing the derivative with respect to ω^2 we have that the minimum is achieved for $\omega^2 = 2$. This in turn yields $\min \mathrm{Re}[P(j\omega)] = -\frac{1}{3}$. Hence, the maximal gain allowed is $k = 3$.

4. Exercise

a) Up to a real change of coordinates A can be put in a block-diagonal form, with the diagonal blocks of type:

$$\dot{x} = \lambda x \qquad x \in \mathbb{R}$$

for a real eigenvalue in λ, or:

$$\dot{x} = \begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix} x \qquad x \in \mathbb{R}^2$$

for complex conjugate ones in $\lambda \pm j\omega$. In the first case, $V(x) = x^2$ provides the desired estimate; indeed:

$$\dot{V}(x) = 2x\dot{x} = 2\lambda x^2 \leq 2\lambda_{\max} V(x)$$

Similarly, in the case of complex conjugate eigenvalues we let $V(x) = x'x$:

$$\dot{V} = x'(A' + A)x = 2\lambda x'x = 2\lambda_{\max} V(x)$$

Hence, any linear combination of such functions works as a suitable Lyapunov function for the overall block-diagonal system. The inequality preserve their validity in original coordinates.

b) Let A be given by:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly $\lambda_{\max} = 0$. So, existence of P as requested yields for $V(x) = x'Px$

$$\dot{V} = x'(A'P + PA)x \leq 0$$

This in turn implies $V(x(t)) \leq V(x(0))$ and hence, boundedness of solutions. However, it is well known that the above system admits t as one of its modes, that is, admits unbounded solutions. This provides the sought contradiction.

c) In fact, the differential inequality that we proved yields:

$$V(x(t)) \leq e^{2\lambda_{\max} t} V(x(0))$$

In particular then, if $\lambda_{\max} < 0$ this provides a proof of exponential stability, since:

$$\underline{\sigma}(P)\|x(t)\|^2 \leq V(x(t)) \leq e^{2\lambda_{\max} t} V(x(0)) \leq e^{2\lambda_{\max} t} \bar{\sigma}(P)\|x(0)\|^2$$

5. Exercise

a) A system is said to be passive if, for all input output pairs y, u there exists some M so that it holds

$$\int_0^{+\infty} y(t)u(t)dt \geq M$$

(meaning that $\liminf_{T\to+\infty} \int_0^T y(t)u(t)dt \geq M$, as the above integral need not exist).

b) For the parallel interconnection we have:

$$u = u_1 = u_2 \qquad y = y_1 + y_2$$

Hence,

$$\int_0^{+\infty} y(t)u(t)dt = \int_0^{+\infty} (y_1(t) + y_2(t))u(t)dt$$
$$\geq \int_0^{+\infty} y_1(t)u_1(t)dt + \int_0^{+\infty} y_2(t)u_2(t) \geq M_1 + M_2.$$

For the feedback interconnection we have:

$$u = u_1 + y_2 \qquad y = y_1 = u_2$$

Hence:

$$\int_0^{+\infty} y(t)u(t)dt = \int_0^{+\infty} y(t)(u_1(t) + y_2(t))dt$$
$$\geq \int_0^{+\infty} y_1(t)u_1(t)dt + \int_0^{+\infty} y_2(t)u_2(t)dt \geq M_1 + M_2,$$

which again shows passivity.

c) Consider the series interconnection of two copies of the following elementary system:

$$\dot{x} = -x + u \qquad y = x$$

The transfer function of the series is $G(s) = \frac{1}{(s+1)^2}$. For all $\omega > 1$, we have $\text{Arg}[G(j\omega)] < -\pi/2$. Hence the systems violates the frequency-domain characterization of passivity. Any sinusoidal input with frequency larger than 1, besides, violates the passivity definition. Indeed:

$$\int_0^{2\pi/\omega} \sin(\omega t)\sin(\omega t + \phi)dt = \frac{1}{2}\int_0^{2\pi/\omega} \cos(\phi) - \cos(2\omega t + \phi)dt = \frac{\pi}{\omega}\cos(\phi)$$

The latter is a negative quantity whenever the phase-lag introduced by the system is larger than $\pi/2$. Hence, for such ωs,

$$\liminf_{T\to+\infty} \int_0^T y(t)u(t)dt \leq \lim_{k\to+\infty} \int_0^{k2\pi/\omega} y(t)u(t)dt = k\frac{\pi}{\omega}\cos(\phi) = -\infty$$

which violates passivity definition.

d) Consider next the nonlinear resistor:

$$VI = R(I)I$$

If $R(I)I \geq 0$ for all I, we have:

$$\int_0^t V(t)I(t)dt \geq 0$$

for all t For the nonlinear inductor:

$$VI = IL(I)\dot{I} = \frac{d}{dt}\int_0^I iL(i)di$$

Hence, if $L(i) \geq \varepsilon > 0$ for all i, the function $E(I) \doteq \int_0^I iL(i)di$ is positive semidefinite and:

$$\int_0^t V(t)I(t)dt = E(I(t)) - E(I(0)) \geq -E(I(0)) > -\infty$$

Similarly for the nonlinear capacitor:

$$VI = VC(V)\dot{V} = \frac{d}{dt}\int_0^V vC(v)dv$$

Hence, if $C(v) \geq \varepsilon > 0$ for all v the function $E(V) \doteq \int_0^V vC(v)dv$ is positive semidefinite and:

$$\int_0^t V(t)I(t)dt = E(V(t)) - E(V(0)) \geq -E(V(0))$$

e) The series interconnection (in the sense of circuit theory) of an inductor and a capacitor is characterized by the following equations:

$$V = V_C + V_L \qquad I = I_C = I_L$$

These are exactly the equations which characterize the feedback interconnection of two systems, taking, respectively $u = I_C$ and $y = V_C$ for the capacitor and $u = V_L$ and $y = I_L$ for the inductor. By the above considerations nonlinear capacitors and inductors correspond to lossless elements. The second order system (with input V and output I) arising from their feedback interconnection is therefore a lossless system.

6. Exercise

a) It is assumed $y = x_1$. Hence deriving the output 3 times we obtain:

$$\begin{aligned}
\dot{y} &= -k\sin(x_1) + x_2 \\
\ddot{y} &= -k\cos(x_1)[-k\sin(x_1) + x_2] + \text{atn}(x_2) + x_3 \\
&= \frac{1}{2}k^2\sin(2x_1) - k\cos(x_1)x_2 + \text{atn}(x_2) + x_3 \\
y^{(3)} &= [k^2\cos(2x_1) + k\sin(x_1)][-k\sin(x_1) + x_2] \\
&\quad + \left[\frac{1}{1+x_2^2} - k\cos(x_1)\right] \cdot [\text{atn}(x_2) + x_3] + u
\end{aligned}$$

Since u only appears at the third derivative, the relative degree is 3.

b) Moreover, the coefficient of u is constant (and different from 0), hence it is possible to globally feedback linearize the system.

c) A globally stabilizing control law is:

$$\begin{aligned}
u = &-[k^2\cos(2x_1) + k\sin(x_1)][-k\sin(x_1) + x_2] \\
&-\left[\tfrac{1}{1+x_2^2} - k\cos(x_1)\right] \cdot [\text{atn}(x_2) + x_3] - y - 3\dot{y} - 3\ddot{y}
\end{aligned}$$

Under such feedback the equations read $y^{(3)} + 3\ddot{y} + 3\dot{y} + y = 0$, which is a linear system with 3 eigenvalues in -1.

d) Let us fix $y = x_2$. This choice gives:

$$\begin{aligned} \dot{y} &= \text{atn}\,(x_2) + x_3 \\ \ddot{y} &= \frac{\text{atn}(x_2) + x_3}{1 + x_2^2} + u \end{aligned}$$

Hence, the relative degree is 2. The system is globally input-output feedback linearizable, however there are non-empty zero-dynamics. In particular the x_1-equation is the zero-dynamics. For $x_2 = 0$ the zero dynamics have infinitely many equilibria at $x_1 = n\pi$ for all $n \in \mathbb{Z}$. Hence the zero-dynamics are not globally asymptotically stable (and not Input-to-State Stable). If $k > 0$ they are locally asymptotically stable at the origin (easy to see by linearization). Hence, a local feedback stabilizer can be obtained by letting:

$$u = -\frac{\text{atn}(x_2) + x_3}{1 + x_2^2} - y - 2\dot{y}.$$

Notice that the above feedback does not assume knowledge of k; it is, however, only guaranteed to converge locally.

e) We now proceed to design a robust feedback stabilizer by means of backstepping. Consider the x_1 equation. This is ISS stabilized (with respect to actuators disturbances and regarding x_2 as an input), by applying the virtual control

$$x_2^v = -2\varepsilon x_1$$

This is easily seen taking $x_1^2/2$ as a Lyapunov function and exploiting $k \in [-\varepsilon, +\varepsilon]$.

Next we consider the (x_1, x_2) subsystem and try to ISS stabilize it by means of the virtual input x_3. To this end we pick the Lyapunov function:

$$V(x_1, x_2) = \frac{x_1^2 + \alpha(x_2 - x_2^v)^2}{2}$$

Taking derivatives yields:

$$\begin{aligned} \dot{V} &\doteq -k\sin(x_1)x_1 + x_1x_2 + \alpha(x_2 + 2\varepsilon x_1)[\text{atn}(x_2) + x_3 - 2\varepsilon k\sin(x_1) + 2\varepsilon x_2] \\ &= -k\sin(x_1)x_1 - 2\varepsilon x_1^2 + (x_2 + 2\varepsilon x_1)[x_1 + \alpha\text{atn}(x_2) + \alpha x_3 - 2\alpha\varepsilon k\sin(x_1) + 2\alpha\varepsilon x_2] \\ &\leq -\varepsilon x_1^2 + x_1(1 + 2\alpha\varepsilon^2]|x_2 + 2\varepsilon x_1| + (x_2 + 2\varepsilon x_1)[\alpha\text{atn}(x_2) + \alpha x_3 + 2\alpha\varepsilon x_2] \end{aligned}$$

Hence, the (x_1, x_2) subsystem is ISS stabilized by picking

$$x_3^v = -\text{atn}(x_2) - 2\varepsilon x_2 - \gamma(x_2 + 2\varepsilon x_1)$$

provided γ is picked sufficiently large, for instance:

$$\gamma > \frac{(1/2 + \alpha\varepsilon^2)^2}{\varepsilon\alpha}.$$

The last step is to backstep x_3^v. We use the Lyapunov function:

$$W(x_1, x_2, x_3) \doteq V(x_1, x_2) + \frac{\beta}{2}(x_3 - x_3^v)^2$$

Taking derivatives of W gives a term proportional to $k\sin(x_1)(x_3 - x_3^v)$. Since k is not known we need to dominate this by introducing a sufficiently large term: $-\delta(x_3 - x_3^v)$ in our control law.