

LINEAR OPTIMAL CONTROL

1. Consider the following finite-horizon discrete-time linear quadratic regulator problem:

$$\pi^*(x_0) := \arg \min_{\pi} x'_N Q x_N + \sum_{k=0}^{N-1} (x'_k Q x_k + u'_k R u_k)$$

where the system dynamics is given by

$$x_{k+1} = Ax_k + Bu_k,$$

and the policy $\pi := \{\mu_0, \mu_1, \dots, \mu_{N-1}\}$ defines the state feedback control law

$$u_k = \mu_k(x_k).$$

The matrices Q and R are assumed to be symmetric.

- a) Show that the optimal control policy is given by

$$\mu_k^*(x_k) = -(B'P_{k+1}B + R)^{-1}B'P_{k+1}Ax_k$$

where P_k is given by

$$P_k = A'(P_{k+1} - P_{k+1}B(B'P_{k+1}B + R)^{-1}B'P_{k+1})A + Q$$

with boundary condition $P_N = Q$.

State any additional assumptions you have made and the reasons for making these additional assumptions. [16]

- b) How would you modify your assumptions and results if Q and R were not symmetric? Justify your answer. [4]

2. Consider the following scalar discrete-time system:

$$x_{k+1} = ax_k + u_k + w_k$$

where $a \in \mathbb{R}$, $x_k \in \mathbb{R}$ is the system state, $u_k \in \mathbb{R}$ is the control input and $w_k \in \mathbb{W}$ is an unmeasurable disturbance that satisfies

$$-1 \leq w_k \leq 1, \quad k = 0, 1, \dots$$

Consider the design of a state feedback gain $L \in \mathbb{R}$ such that

$$u_k = Lx_k, \quad k = 0, 1, \dots$$

In the following, assume that the initial state $x_0 = 0$.

a) Show that, if c is any given scalar, then

$$\max_{-1 \leq w_k \leq 1} cw_k = |c|$$

[4]

b) Show that, for the closed-loop system,

$$s_k := \max_{w_0, w_1, \dots, w_{k-1}} |x_k| = \max_{w_0, w_1, \dots, w_{k-1}} x_k, \quad k = 1, 2, \dots,$$

where the constraints on the disturbance sequence $\{w_0, \dots, w_{k-1}\}$ are as above. Hence, show that the sequence of the maximum magnitude of the state, i.e. $\{s_1, s_1, s_2, \dots\}$, is non-decreasing and that the state trajectory of the closed-loop system is bounded if and only if $|a + L| < 1$. [8]

c) Compute the feedback gain L that minimizes the maximum deviation of the state from the origin over all time, i.e. compute the gain that solves the following optimal control problem:

$$L^* := \arg \min_L \left\{ \max_{k=0,1,\dots} s_k \right\}.$$

Hint: You may wish to use the fact that $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ if and only if $|r| < 1$. [8]

3. a) Consider the following continuous-time optimal control problem:

$$v^*(\cdot) = \arg \min_{v(\cdot)} \int_0^{\infty} (z(t)' \bar{Q} z(t) + v(t)' \bar{R} v(t)) dt,$$

where \bar{Q} and \bar{R} are constant (time-invariant) matrices and the continuous-time dynamics are given by

$$\dot{z} = \bar{A}z + \bar{B}v.$$

From standard LQR theory it can be shown that the optimal control law is given by

$$v^*(t) = \bar{L}z(t) = -\bar{R}^{-1} \bar{B}' \bar{P} z(t),$$

where \bar{P} is the positive semidefinite solution of the continuous-time algebraic Riccati equation (ARE)

$$\bar{A}' \bar{P} + \bar{P} \bar{A} + \bar{Q} - \bar{P} \bar{B} \bar{R}^{-1} \bar{B}' \bar{P} = 0.$$

Give conditions on \bar{Q} , \bar{R} , \bar{A} and \bar{B} which guarantee that the ARE has a unique stabilizing solution. [4]

- b) It is sometimes desired to have all the eigenvalues of a closed-loop system with real parts less than some negative number $-\alpha$, $\alpha > 0$. This is commonly referred to as “degree of stability $-\alpha$ ”. It turns out that this is simple to design with LQR theory, given suitable time-varying choices of state penalty $Q(t)$ and input penalty $R(t)$.

In particular, consider now the continuous-time optimal control problem:

$$u^*(\cdot) := \arg \min_{u(\cdot)} \int_0^{\infty} (x(t)' Q(t) x(t) + u(t)' R(t) u(t)) dt,$$

where the continuous-time dynamics are given by

$$\dot{x} = Ax + Bu.$$

Show that if

$$Q(t) := e^{2\alpha t} \bar{Q} \text{ and } R(t) := e^{2\alpha t} \bar{R},$$

then the optimal LQR control law, which guarantees that the closed-loop system has “degree of stability $-\alpha$ ”, is given by

$$u^*(t) = Lx(t) = -\bar{R}^{-1} \bar{B}' \bar{P} x(t)$$

where \bar{P} satisfies

$$(A + \alpha I)' \bar{P} + \bar{P} (A + \alpha I) + \bar{Q} - \bar{P} \bar{B} \bar{R}^{-1} \bar{B}' \bar{P} = 0.$$

Hint: If one can show that $x(t) = e^{-\alpha t} z(t)$, where $z(t)$ is the state of an asymptotically stable system, then the closed-loop system $\dot{x} = (A + BL)x$ has “degree of stability $-\alpha$ ”. [16]

4. Consider the discrete-time optimal control problem:

$$\pi^*(x_0) := \arg \min_{\pi} \mathbb{E} \left\{ c'x_N + \sum_{k=0}^{N-1} [c'x_k + d(u_k)] \right\}$$

where the system dynamics are given by

$$x_{k+1} = A_k x_k + \beta(u_k) + w_k,$$

and the policy $\pi := \{\mu_0, \mu_1, \dots, \mu_{N-1}\}$ defines the state feedback control law

$$u_k = \mu_k(x_k).$$

In the above, c is a given vector, $d(\cdot)$ and $\beta(\cdot)$ are given nonlinear functions, A_k and w_k are random $n \times n$ matrices and n -dimensional vectors, respectively, with given probability distributions that do not depend on x_k , u_k or prior values of A_k and w_k . The expectation $\mathbb{E}\{\cdot\}$ is taken with respect to A_k and w_k , $k = 0, 1, \dots, N-1$.

- a) State the “principle of optimality” in words. [4]
- b) Show that the cost-to-go functions of the Dynamic Programming algorithm for the above optimal control problem are affine (linear plus a constant). [10]
- c) What is meant with “certainty equivalence”? Determine whether or not “certainty equivalence” holds for the above optimal control problem. [6]

5. a) Consider the quadratic form

$$\ell(z, u) := z'Qz + u'Ru + 2u'Sz,$$

where R is positive definite. Show that $\ell(z, u) \geq 0$ for all (z, u) if and only if $Q - S'R^{-1}S$ is positive semidefinite.

Hint: You may wish to use the fact that $\ell(z, u) \geq 0$ for all (z, u) if and only if the function $L(z) := \min_u \ell(z, u) \geq 0$ for all z . [6]

- b) A popular cost function, especially in predictive control applications, is the following:

$$J := y'_N M y_N + \sum_{k=0}^{N-1} (y'_k M y_k + u'_k V u_k + (\Delta u_k)' W \Delta u_k),$$

where M , V and W are symmetric matrices, the discrete-time dynamics are given by

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k,$$

and the change in control input at time k is defined as

$$\Delta u_k := u_k - u_{k-1}.$$

Show that, by defining the augmented state vector

$$z_k := [x'_k \ u'_{k-1}]',$$

one can rewrite the cost function in the form

$$J = z'_N Q z_N + \sum_{k=0}^{N-1} (z'_k Q z_k + u'_k R u_k + 2u'_k S z_k)$$

where the augmented discrete-time dynamics are given by

$$z_{k+1} = \bar{A}z_k + \bar{B}u_k$$

with \bar{A} , \bar{B} , Q , R and S suitably defined. [10]

- c) Give sufficient conditions on M , V and W such that R is positive definite and $Q - S'R^{-1}S$ is positive semi-definite, with Q , R and S as in part b). [4]

6. From standard LQR theory, it can be shown that the solution to the problem

$$u^*(\cdot) := \arg \min_{u(\cdot)} \int_0^{\infty} (x(t)'Qx(t) + u(t)'Ru(t)) dt,$$

where the continuous-time dynamics are given by

$$\dot{x} = Ax + Bu,$$

is given by

$$u^*(t) = -R^{-1}B'Px(t),$$

where P satisfies the continuous-time algebraic Riccati equation (ARE)

$$A'P + PA + Q - PBR^{-1}B'P = 0.$$

The linearized continuous-time state space model for an inverted pendulum is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \gamma & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

where $\gamma > 0$ is a given scalar. A control law is sought to minimize the performance index

$$\int_0^{\infty} \left(x_1(t)^2 + \frac{u(t)^2}{c} \right) dt,$$

where $c > 0$ is a given scalar.

- a) Show that a stabilizing solution exists to the above LQR problem for the inverted pendulum. [6]
- b) Show that the optimal LQR control law is given by

$$u(t) = - \left[\gamma + \sqrt{\gamma^2 + c} \quad \sqrt{2(\gamma + \sqrt{\gamma^2 + c})} \right] x(t).$$

[14]