

# Model Answers for E410 Probability & Stochastic Process

Q 1:

a) i)  $P(\text{function after } t \text{ sec})$  E 4.10  
C 2.1  
SL 4

$$= P(\text{function after } t \text{ sec} \mid \text{bad device}) \cdot P(\text{bad device})$$

$$+ P(\text{function after } t \text{ sec} \mid \text{good device}) \cdot P(\text{good device})$$

$$\Rightarrow P(\text{function after } t \text{ sec}) = [1 - F_b(t)] \cdot (1-p)$$

$$+ [1 - F_g(t)] \cdot p$$

ii) Let  $G$  be the event that a device is "good"  
 $B$  be the event that a device is "bad"  
 $F$  be the event that a device still functions

$$P(G|F) = 0.99$$

By Bayes' rule:

$$P(G|F) = \frac{P(G \cap F)}{P(F)}$$

$$\Rightarrow P(G|F) = \frac{P(F|G)P(G)}{P(F|G)P(G) + P(F|B)P(B)}$$

$$\Rightarrow P(G|F) = \frac{(1 - F_g(t)) \cdot p}{(1 - F_g(t)) \cdot p + (1 - F_b(t)) \cdot (1-p)}$$

$$\Rightarrow \frac{(1 - F_g(t)) \cdot p}{(1 - F_g(t)) \cdot p + (1 - F_b(t)) \cdot (1-p)} = 0.99$$

Q1

a) iii) For exponential distribution,

$$F_g(t) = 1 - e^{-\lambda t}$$

$$F_b(t) = 1 - e^{-1000\lambda t}$$

$$\Rightarrow \frac{pe^{-\lambda t}}{pe^{-\lambda t} + (1-p)e^{-1000\lambda t}} = 0.99$$

$$\Rightarrow 1 + \frac{1-p}{p} \cdot e^{-999\lambda t} = \frac{100}{99}$$

$$\Rightarrow \frac{1-p}{p} e^{-999\lambda t} = \frac{1}{99}$$

$$\Rightarrow e^{-999\lambda t} = \frac{p}{99(1-p)}$$

$$\Rightarrow -999\lambda t = \ln\left(\frac{p}{99(1-p)}\right)$$

$$t = \frac{-1}{999\lambda} \ln\left(\frac{p}{99(1-p)}\right)$$

$$t = \frac{1}{999\lambda} \ln\left(\frac{99(1-p)}{p}\right)$$

Q1

b) i)  $Y = kX$

$$F_Y(y) = P(Y \leq y) = P(kX \leq y) = P\left(X \leq \frac{y}{k}\right)$$

$$\Rightarrow F_Y(y) = F_X\left(\frac{y}{k}\right)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X\left(\frac{y}{k}\right)}{dy}$$

$$\Rightarrow f_Y(y) = \frac{dF_X(y')}{dy'} \cdot \frac{d\left(\frac{y}{k}\right)}{dy}$$

$$\Rightarrow f_Y(y) = f_X\left(\frac{y}{k}\right) \cdot \frac{1}{k}$$

ii)  $F_Y^*(s) = E[e^{-sY}] = E[e^{-skX}]$

$$\Rightarrow F_Y^*(s) = F_X^*(sk)$$

Q 2

a)

$$G^*(z) = \sum_{k=0}^{\infty} p_k z^k = \sum_{k=0}^{\infty} \frac{(\alpha z)^k}{k!} e^{-\alpha}$$

$$\Rightarrow G^*(z) = e^{-\alpha} \cdot e^{\alpha z} = e^{-\alpha(1-z)}$$

$$E(x) = \left. \frac{dG^*(z)}{dz} \right|_{z=1} = e^{-\alpha} e^{\alpha z} \cdot \alpha \Big|_{z=1} = \alpha$$

$$\left. \frac{d^2 G^*(z)}{dz^2} \right|_{z=1} = E(x^2 - x) = E(x^2) - E(x)$$

$$\left. \frac{d^2 G^*(z)}{dz^2} \right|_{z=1} = e^{-\alpha} e^{\alpha z} \alpha^2 \Big|_{z=1} = \alpha^2$$

$$\Rightarrow \alpha^2 = E(x^2) - \alpha$$

$$\Rightarrow E(x^2) = \alpha^2 + \alpha$$

$$\text{VAR}(x) = E(x^2) - (E(x))^2 = \alpha^2 + \alpha - \alpha^2 = \alpha.$$

Q 2

b. Let  $N_k$  be the number of arrivals during time  $t$  from the  $k^{\text{th}}$  process and  $N = N_1 + N_2 + \dots + N_k$ .

Consider  $K=2$ . Let  $N = N_1 + N_2$ .

$$\begin{aligned} \text{Then } P(N_1 + N_2 = n) &= \sum_{j=0}^n P(N_1 = j) P(N_2 = n-j) \\ &= \sum_{j=0}^n \frac{(\lambda_1 t)^j}{j!} e^{-\lambda_1 t} \cdot \frac{(\lambda_2 t)^{n-j}}{(n-j)!} e^{-\lambda_2 t} \end{aligned}$$

Let  $\lambda = \lambda_1 + \lambda_2$ .

$$\begin{aligned} \Rightarrow P(N_1 + N_2 = n) &= e^{-\lambda t} \cdot \frac{t^n}{n!} \sum_{j=0}^n \binom{n}{j} \lambda_1^j \lambda_2^{n-j} \\ &= e^{-\lambda t} \cdot \frac{t^n}{n!} (\lambda_1 + \lambda_2)^n \end{aligned}$$

$$\Rightarrow P(N_1 + N_2 = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

The result for general  $K$  follows by induction.

Q2

c) Markov inequality

i)

$$P(X \geq a) \leq \frac{E(X)}{a}$$

$$\Rightarrow P(e^{sX} \geq \alpha) \leq \frac{E(e^{sX})}{\alpha} = \frac{\phi(s)}{\alpha}$$

ii) Set  $\alpha = e^{sA}$  in part i.

$$\text{We have } P(e^{sX} \geq e^{sA}) \leq \frac{\phi(s)}{e^{sA}}$$

$$\Rightarrow P(e^{sX} \geq e^{sA}) \leq e^{-sA} \phi(s)$$

Since  $s > 0$ ,

$$P(e^{sX} \geq e^{sA}) = P(X \geq A).$$

$$\text{Therefore } P(X \geq A) \leq e^{-sA} \phi(s)$$

Q3

a) i)

$$u = xy$$

$$v = x$$

The system  $u = xy$  and  $v = x$  has one single solution:  $x = v$  and  $y = u/v$ .

The Jacobian is

$$|J(x, y)| = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -v$$

Therefore,  $f_{uv}(u, v) = \frac{1}{|v|} f_{xy}(v, \frac{u}{v})$

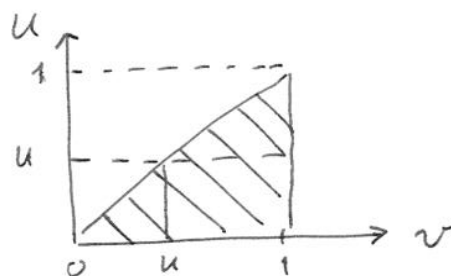
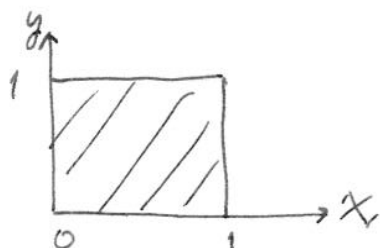
ii) The marginal pdf for  $u$  is

$$f_u(u) = \int_{-\infty}^{\infty} \frac{1}{|v|} \cdot f_{xy}(v, \frac{u}{v}) dv$$

iii) Given that  $X, Y$  are independent and each is uniformly distributed in  $(0, 1)$ . Thus,

$$f_{xy}(v, \frac{u}{v}) = f_x(v) \cdot f_y(\frac{u}{v}) = 1$$

for  $u < v < 1$  and  $0 < u < 1$ , and 0 otherwise.



Q3

a) ii). Subs.  $f_{xy}(u, \frac{u}{v})$  into the result in part ii,  
we obtain

$$f_u(u) = \int_u^1 \frac{1}{v} dv = \begin{cases} -\ln u & 0 < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

b). i) Let the mean square error be  $f$  where

$$f = E[(X(n) - aX(n-2) - bX(n-1))^2]$$

$$\frac{\partial f}{\partial a} = 2 E[(X(n) - aX(n-2) - bX(n-1)) X(n-2)]$$

Set  $\frac{\partial f}{\partial a} = 0$  for minimizing  $f$ . We obtain

$$E[X(n)X(n-2)] - a E[X(n-2)X(n-2)] - b E[X(n-1)X(n-2)] = 0$$

$$\Rightarrow R(2) - aR(0) - bR(1) = 0$$

Given  $R(\tau) = e^{-\tau^2}$ , the above equation becomes

$$e^{-4} - a - e^{-1}b = 0$$

$$\Rightarrow a + e^{-1}b = e^{-4} \quad \text{--- (1)}$$

Q3

b). i) Similarly, setting  $\frac{\partial f}{\partial b} = 0$  leads to

$$E[(X(n) - aX(n-2) - bX(n-1)) \cdot X(n-1)] = 0$$

$$\Rightarrow E[X(n)X(n-1)] - aE[X(n-2) \cdot X(n-1)] - bE[X(n-1) \cdot X(n-1)] = 0$$

$$\Rightarrow R(1) - aR(1) - bR(0) = 0$$

$$\Rightarrow aR(1) + bR(0) = R(1)$$

$$\Rightarrow e^{-1}a + b = e^{-1} \quad \text{--- (2)}$$

Solving ① & ② yields

$$a = -e^{-2} \quad \text{and} \quad b = e^{-1}(1 + e^{-2})$$

Q3

b.) ii.) The mean square error is

$$\begin{aligned}
 f &= E \left[ (X(n) - aX(n-2) - bX(n-1))^2 \right] \\
 &= E \left[ X^2(n) + a^2 X^2(n-2) + b^2 X^2(n-1) \right. \\
 &\quad \left. - 2a X(n)X(n-2) + 2ab X(n-2)X(n-1) \right. \\
 &\quad \left. - 2b X(n)X(n-1) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= R(0) + a^2 R(0) + b^2 R(0) - 2a R(2) \\
 &\quad + 2ab R(1) - 2b R(1)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow f &= (1 + a^2 + b^2)R(0) - 2a R(2) \\
 &\quad + 2b R(1)(a - 1)
 \end{aligned}$$

$$\begin{aligned}
 &= \left( 1 + e^{-4} + e^{-2}(1 + e^{-2})^2 \right) + 2e^{-2}e^{-4} \\
 &\quad + 2 \cdot e^{-1}(1 + e^{-2}) \cdot e^{-1}(-e^{-2} - 1)
 \end{aligned}$$

$$\begin{aligned}
 &= 1 + e^{-4} + e^{-2}(1 + 2e^{-2} + e^{-4}) + 2e^{-6} \\
 &\quad - 2e^{-2}(1 + 2e^{-2} + e^{-4})
 \end{aligned}$$

$$= 1 + e^{-4} + 2e^{-6} - e^{-2}(1 + 2e^{-2} + e^{-4})$$

$$= 1 + e^{-4} + 2e^{-6} - e^{-2} - 2e^{-4} - e^{-6}$$

$$= 1 - e^{-2} - e^{-4} + e^{-6}$$

Q3

ii)  $\Rightarrow$  Mean square error

$$f = (1 - e^{-2})(1 - e^{-4})$$

Q 4

a)  $\sigma_{X(t)}^2 = E(X^2(t))$  since  $E(X(t)) = 0$

$$\Rightarrow \sigma_{X(t)}^2 = R(0) = 1.$$

b) Since  $X(t)$  has a normal distribution, we have

$$P\left(\frac{X(t) - E(X(t))}{\sigma_{X(t)}} \leq y\right) = F(y)$$

As  $E(X(t)) = 0$  and  $\sigma_{X(t)} = 1$ , thus

$$P(X(t) \leq y) = F(y).$$

That is,  $P(X(t) \leq 2) = F(2)$

c). 
$$E[X(t+\tau) + X(t)] = E[X(t+\tau)] + E[X(t)]$$
$$= 0 \quad \because X(t) \text{ is WSS}$$

d) 
$$E[(X(t+\tau) + X(t))^2]$$
$$= E[X^2(t+\tau) + X^2(t) + 2X(t+\tau)X(t)]$$
$$= R(0) + R(0) + 2R(\tau) \quad \because X(t) \text{ is WSS}$$

$$\Rightarrow E[(X(t+\tau) + X(t))^2] = 2R(0) + 2R(\tau)$$
$$= 2(1 + e^{-2|\tau|}).$$

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e.)

Let  $X = X(t)$ ,  $Y = X(t+\tau)$ Define  $S \equiv X+Y = X(t) + X(t+\tau)$  $\rho \equiv R(\tau)$ 

Since  $X$  and  $Y$  are jointly normal distributed, the pdf for  $S$

$$f_S(s) = \int_{-\infty}^{\infty} f_{XY}(x, s-x) dx$$

Subs. the normal pdf for  $X, Y$  into the above, we

get

$$f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[ x^2 - 2\rho x(s-x) + (s-x)^2 \right] \right\} dx$$

Consider the expression inside the brackets

$$\begin{aligned} & x^2 - 2\rho x(s-x) + (s-x)^2 \\ &= x^2 + 2\rho x^2 - 2\rho s x + s^2 - 2xs + x^2 \\ &= 2(1+\rho)x^2 - 2(1+\rho)sx + s^2 \end{aligned}$$

(4.1)

Subs. this into (4.1)

$$f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{1-\rho} [x^2 - sx] - \frac{s^2}{2(1-\rho^2)} \right\} dx$$

$$\Rightarrow f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{1-\rho} \left(x - \frac{s}{2}\right)^2 + \frac{s^2}{4(1-\rho)} - \frac{s^2}{2(1-\rho^2)} \right\} dx$$

$$\Rightarrow f_S(s) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp\left\{ \frac{-1}{2} \cdot \frac{(x-s/2)^2}{(1-\rho)/2} - \frac{s^2}{4(1+\rho)} \right\} dx$$

$$\Rightarrow f_S(s) = \frac{1}{\sqrt{2\pi} \sqrt{1+\rho} \sqrt{2}} \exp\left\{ \frac{-s^2}{4(1+\rho)} \right\} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sqrt{1-\rho}/\sqrt{2}} \cdot \exp\left\{ \frac{-1}{2} \frac{(x-s/2)^2}{(1-\rho)/2} \right\} dx}_{=1}$$

$$\Rightarrow f_S(s) = \frac{1}{\sqrt{2\pi} \sqrt{1+\rho} \sqrt{2}} \cdot \exp\left\{ \frac{-s^2}{4(1+\rho)} \right\}$$

is a normal pdf

That is,  $S^t = X(t+t) + X(t)$  is normally distributed.

Q4  
f.)

$$\text{Let } S = X(t+\tau) + X(t)$$

$$P(|S| \leq 1) = P\left(\frac{|S - E(S)|}{\sigma_S} \leq \frac{1}{\sigma_S}\right)$$

$\therefore E(S) = 0$  from part c and  $\sigma_S > 0$   
which has been obtained in part d.

$$\Rightarrow P(|S| \leq 1) = P\left(\frac{|S - E(S)|}{\sigma_S} \leq \frac{1}{\sigma_S}\right) = F\left(\frac{1}{\sigma_S}\right) - F\left(\frac{-1}{\sigma_S}\right)$$

$$\text{where } \sigma_S = [2(1 + e^{-2|\tau|})]^{1/2}$$

from part d.

Q5.

a.) i)  $E(S_n) = n \cdot m$

$$\text{VAR}(S_n) = n \cdot m^2$$

ii)  $P(950 \text{ m} < S_{1000} \leq 1050 \text{ m})$

$$= P\left(\frac{950 \text{ m} - 1000 \text{ m}}{m \sqrt{1000}} \leq Z_{1000} \leq \frac{1050 \text{ m} - 1000 \text{ m}}{m \sqrt{1000}}\right)$$

by Central Limit Theorem

$$\Rightarrow P(950 \text{ m} < S_{1000} \leq 1050 \text{ m})$$

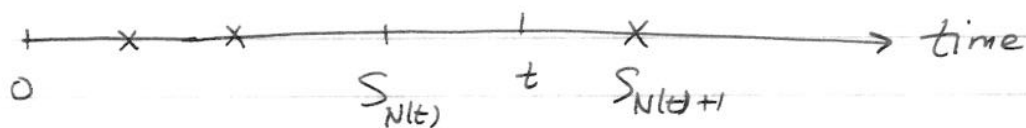
$$\doteq Q(1.58) - Q(-1.58)$$

$$= 1 - 2Q(1.58)$$

$$= 0.8866$$

Q5

b.) iii)



Since  $S_{N(t)} \leq t < S_{N(t)+1}$ , we have

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} \quad \text{--- (S.1)}$$

By the law of Large Number

$$\frac{S_{N(t)}}{N(t)} \rightarrow \mu \quad \text{as } t \rightarrow \infty \quad \text{--- (S.2)}$$

Now, we write that

$$\frac{S_{N(t)+1}}{N(t)} = \left[ \frac{S_{N(t)+1}}{N(t)+1} \right] \cdot \left[ \frac{N(t)+1}{N(t)} \right]$$

By the same argument of the <sup>law of</sup> Large Number,

$$\frac{S_{N(t)+1}}{N(t)+1} \rightarrow \mu \quad \text{as } t \rightarrow \infty$$

Further,  $\frac{N(t)+1}{N(t)} \rightarrow 1 \quad \text{as } t \rightarrow \infty$

Therefore,  $\frac{S_{N(t)+1}}{N(t)} \rightarrow \mu \quad \text{as } t \rightarrow \infty \quad \text{--- (S.3)}$

Q5

b.) i)

Note that  $N(t) \geq n \iff S_n \leq t$ 

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \end{aligned}$$

Since  $X_i$ 's are iid with a common PDF  $F(x)$ ,the PDF for  $S_n = \sum_{i=1}^n X_i$  is the  $n$ -foldconvolution of  $F(t)$  with itself.

Thus, 
$$P(N(t) = n) = F_n(t) - F_{n+1}(t)$$

where  $F_n(t) \stackrel{\#}{=} n$ -fold convolution of  $F(t)$ 

ii) Define 
$$N(t) = \sum_{n=1}^{\infty} I_n$$

where 
$$I_n = \begin{cases} 1 & \text{if the } n^{\text{th}} \text{ arrival} \\ & \text{occurs in } (0, t] \\ 0 & \text{otherwise} \end{cases}$$

Therefore 
$$E[N(t)] = E\left[\sum_{n=1}^{\infty} I_n\right]$$

$$= \sum_{n=1}^{\infty} E[I_n]$$

$$= \sum_{n=1}^{\infty} P(I_n = 1)$$

$$= \sum_{n=1}^{\infty} P(S_n \leq t)$$

$$\Rightarrow E[N(t)] = \sum_{n=1}^{\infty} F_n(t)$$

Q5.

b) iii)

Putting (5.2) & (5.3) into (5.1),

We must have

$$\frac{t}{N(t)} \rightarrow \mu \text{ as } t \rightarrow \infty \quad \text{Q.E.D.}$$

Q6.

a.) For all  $i = 0, 1, 2, \dots$

$$\pi_i = \sum_{j=0}^{\infty} \pi_j p_{ji}$$

b)

$$Q(z) = \sum_{i=0}^{\infty} \pi_i z^i = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_j p_{ji} z^i$$

Subs.  $p_{ij}$  into the above, we have

$$Q(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_j z^i e^{-\lambda} \sum_{n=0}^i \binom{j}{n} p^n g^{j-n} \frac{\lambda^{i-n}}{(i-n)!}$$

Since  $\sum_{i=0}^{\infty} \sum_{n=0}^i \Leftrightarrow \sum_{n=0}^{\infty} \sum_{i=n}^{\infty}$ , we have

$$\begin{aligned} Q(z) &= e^{-\lambda} \sum_{j=0}^{\infty} \pi_j \sum_{n=0}^{\infty} \binom{j}{n} p^n g^{j-n} \sum_{i=n}^{\infty} z^i \frac{\lambda^{i-n}}{(i-n)!} \\ &= e^{-\lambda} e^{\lambda z} \sum_{j=0}^{\infty} \pi_j \sum_{n=0}^{\infty} \binom{j}{n} (pz + g)^n g^{j-n} \end{aligned}$$

Since  $\binom{j}{n} = 0$  for  $n > j$ , then

$$Q(z) = e^{\lambda(z-1)} \sum_{j=0}^{\infty} \pi_j (pz + g)^j$$

$$\Rightarrow Q(z) = e^{\lambda(z-1)} Q(pz + g)$$

$$\Rightarrow Q(z) = e^{\lambda(z-1)} Q(1 + p(z-1))$$

Q6.  
d.)

$$Q(z) = e^{\lambda(z-1)(1+p+p^2+\dots+p^{n-1})}$$

$$\cdot Q(1+p^n(z-1))$$

When  $n=1$ ,

$$Q(z) = e^{\lambda(z-1)} Q(1+p(z-1))$$

which is valid as part b shows.

By induction, assume it is true that for some arbitrary integer  $n$ ,

$$Q(z) = e^{\lambda(z-1)(1+p+p^2+\dots+p^{n-1})} Q(1+p^n(z-1))$$

Now, substitute  $1+p^n(z-1)$  for  $z$  in the (6.1)

result from part b; we then get

$$Q(1+p^n(z-1)) = e^{-\lambda p^n(z-1)} Q(1+p^{n+1}(z-1))$$

Subs. the above to the RHS of (6.1) gives

$$Q(z) = e^{\lambda(z-1)(1+p+p^2+\dots+p^{n-1}+p^n)} \\ \cdot Q(1+p^{n+1}(z-1))$$

which completes the induction.

Q 6.  
d.)

$$Q(z) = e^{\lambda(z-1)(1+p+p^2+\dots+p^n)} Q(1+p^{n+1}(z-1))$$

for all  $n=0, 1, 2, \dots$ . Consider  $n \rightarrow \infty$ . This

gives

$$Q(z) = e^{\lambda(z-1) \sum_{n=0}^{\infty} p^n} Q(1)$$

as  $p^{n+1} \rightarrow 0$  as  $n \rightarrow \infty \quad \because p < 1$ .

Since  $Q(1) = 1$ , we then have

$$Q(z) = e^{\lambda(z-1) \sum_{n=0}^{\infty} p^n}$$

$$\Rightarrow Q(z) = e^{\lambda(z-1) \cdot \frac{1}{1-p}}$$

$$\text{or } Q(z) = e^{\lambda(z-1)/8} = e^{\frac{\lambda}{8}(z-1)}$$

Expand  $Q(z)$  in terms of  $z$ . We obtain

$$Q(z) = e^{-\frac{\lambda}{8}} \left( 1 + \frac{\lambda}{8}z + \frac{1}{2!} \left(\frac{\lambda}{8}\right)^2 z^2 + \frac{1}{3!} \left(\frac{\lambda}{8}\right)^3 z^3 + \dots \right)$$

$$\text{By inspection, } \pi_j = e^{-\frac{\lambda}{8}} \cdot \frac{(\lambda/8)^j}{j!}$$

for  $\forall j=0, 1, 2, \dots$