Solutions - 2008 E4.08 [Ite 4.33] SO14

1. a) i) **What does the term Singular Value Decomposition (SVD) of an image mean?**

It is the process of expanding the image in terms of vector outer products of the eigenvectors of the matrix created by multiplying the image with its transpose. The coefficients of the expansion are the square roots of the eigenvalues of this matrix.

[5]

ii) **If we truncate the singular value expansion of an image, what is the approximation error?**

The square error is equal to the sum of the omitted eigenvalues.

[5]

iii) **If A is a matrix, show that the eigenvalues of AA^T are always non-negative numbers.**

$$
\begin{aligned}
AA^T\mathbf{x} &= \lambda\mathbf{x} \Rightarrow \\
\mathbf{x}^T AA^T\mathbf{x} &= \mathbf{x}^T\lambda\mathbf{x} \Rightarrow \\
(\mathbf{x}^T A)(A^T\mathbf{x}) &= \lambda \underbrace{\mathbf{x}^T\mathbf{x}}_{=1,\mathbf{x}=eigenvector} \Rightarrow \\
\underbrace{(A^T\mathbf{x})^T}_{\equiv\mathbf{y}^T}\underbrace{(A^T\mathbf{x})}_{\equiv\mathbf{y}} &= \lambda \Rightarrow \\
\lambda &= \mathbf{y}^T\mathbf{y} \geq 0
\end{aligned}
$$

[20]

iv) **If u is an eigenvector of matrix AA^T with eigenvalue λ, work out the eigenvector of A^TA and the corresponding eigenvalue.**

$$
\begin{aligned}
AA^T\mathbf{x} &= \lambda\mathbf{x} \Rightarrow \\
A^TAA^T\mathbf{x} &= A^T\lambda\mathbf{x} \Rightarrow \\
A^TA(A^T\mathbf{x}) &= \lambda(A^T\mathbf{x})
\end{aligned}
$$

So $A^T\mathbf{x}$ is an eigenvector of A^TA with eigenvalue λ.

[20]

b) i) **Perform the SVD of the following image, showing all your workings:**

$$
g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{1.1}
$$

$$
gg^T = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \tag{1.2}
$$

The eigenvalues of this matrix are the solutions of

$$\begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda-4)\lambda = 0 \qquad (1.3)$$

For eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$ we work out the following normalised eigenvectors:

$$\begin{aligned} \mathbf{u_1}^T &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \\ \mathbf{u_2}^T &= (0,1,0) \end{aligned}$$

The SVD of the image therefore is:

$$\begin{aligned} g &= \sqrt{\lambda_1}\mathbf{u_1}\mathbf{u_1}^T + \sqrt{\lambda_2}\mathbf{u_2}\mathbf{u_2}^T \\ &= 2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0,1,0) \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

[50]

2. a) i) **What does "additive, white, Gaussian noise" mean? Explain your answer by considering how a noise value affects the true value of a particular pixel.**

The noise value is added to the true pixel value. The noise value is drawn from a Gaussian probability density function that depends on parameters that are not influenced by the noise values in other pixels in the image.

[10]

ii) **What is the difference between "uncorrelated" and "white" noise? Justify your answer.**

There is no difference. The terms are identical. Uncorrelated noise means that the correlation function of the noise field is a delta function. The Fourier transform of a delta function is a flat function. The Fourier transform of the correlation function is the power spectral density of the noise field, which therefore will be flat, ie white.

[10]

iii) **What type of interference is variable illumination? Justify your answer.**

It is multiplicative interference. The illumination field is multiplied with the reflectance function field to produce the imaged scene.

[10]

iv) **How can we reduce the effect of variable illumination in an image? Include in your answer a drawing of the transfer function of the filter you should use.**

We first convert the multiplicative interference to additive by taking the logarithm of the image. Then we take the DFT of the image and multiply it with a homomorphic filter that suppresses low frequencies and leaves unchanged high frequencies. Then we take the inverse DFT and finally we exponentiate.

[10]

v) **For what type of noise will you use a median filter? Justify your answer with a small example.**

For impulse noise. If the true values of a patch are $2,3,5,3,2,4,5,5,6$, and impulse noise corrupts one of them, the values this patch will contain will be, say, $2,100,5,3,2,4,5,5,6$. The sorted values in the two cases are $2,2,3,3,4,5,5,5,6$ and $2,2,3,4,5,5,5,6,100$. The median of the true values is 4, while of the corrupted values is 5, ie very close to the true median.

[10]

b) **You are given the following image:**

$$g = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 & 2 \\ 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad (2.1)$$

In the two processings you will have to perform next, do not process the border pixels. Present your two answers as two 5×5 images with the border pixels left blank.

i) **Process it with a median filter of size 3×3.**

The answer is:

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (2.2)$$

[20]

ii) **Process it with a low pass filter with weights:**

$$g = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad (2.3)$$

[30]

The answer is:

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 14 & 17 & 22 & 0 \\ 0 & 14 & 14 & 22 & 0 \\ 0 & 14 & 14 & 19 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (2.4)$$

3. a) **The Haar functions are defined as follows:**

$$H_0(t) = 1 \text{for } 0 \le t < 1$$

$$H_1(t) = \begin{cases} 1 & \text{for } 0 \le t < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \le t < 1 \end{cases}$$

$$H_{2^p+n}(t)) = \begin{cases} \sqrt{2}^p & \text{for } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^p & \text{for } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases} \quad (3.1)$$

for $p = 1, 2, 3, \ldots$ **and** $n = 0, 1, \ldots, 2^p - 1$.

Work out the 4×4 **matrix with which you can obtain the Haar transform of a** 4×4 **image.**

Use $p = 1$ and $n = 0$ to get:

$$H_2(t)) = \begin{cases} \sqrt{2} & \text{for } 0 \le t < \frac{1}{4} \\ -\sqrt{2} & \text{for } \frac{1}{4} \le t < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases} \quad (3.2)$$

Use $p = 1$ and $n = 1$ to get:

$$H_3(t)) = \begin{cases} 0 & \text{for } 0 \le t < \frac{1}{2} \\ \sqrt{2} & \text{for } \frac{1}{2} \le t < \frac{3}{4} \\ -\sqrt{2} & \text{for } \frac{3}{4} \le t < 1 \end{cases} \quad (3.3)$$

Multiply the independent variable t with 4 and sample at integer values to create the digitised versions of H_0^T, H_1^T, H_2^T and H_3^T. Write them one under the other to form the transformation matrix:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad (3.4)$$

[50]

b) **Work out the basis images in terms of which the Haar transform expands a** 4×4 **image.**

These are given by taking the vector outer product of every H_i^T with every other. For example, one of them is:

$$H_1 H_0^T = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} (1,1,1,1) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \quad (3.5)$$

Another is:

$$H_1 H_1^T = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} (1, 1, -1, -1) = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \qquad (3.6)$$

There should be in total 9 such products.

[50]

4. a) **You have an image that depicts a bright object on a dark background. You know that the object occupies a fraction θ of the image pixels. You want to separate the object from the background by thresholding the image using a threshold t. Work out the equation that you have to solve in order to identify a value of threshold t that minimised the total number of misclassified pixels.**

Total error:

$$E(t) = \theta \int_{-\infty}^{t} p_o(x)dx + (1-\theta)\int_{t}^{\infty} p_b(x)dx \tag{4.1}$$

Differentiate with respect to t and set to 0 to have an equation for t:

$$\frac{\partial E(t)}{\partial t} = \theta p_o(t) - (1-\theta)p_b(t) = 0 \tag{4.2}$$

[40]

b) **You are told that the grey values of the pixels that make up the object are drawn from probability density function**

$$p_o(x) = \frac{1}{2\sigma_o}\exp\left(-\frac{|x-\mu_o|}{\sigma_o}\right) \tag{4.3}$$

while the grey values of the pixels that make up the background are drawn from probability density function:

$$p_b(x) = \frac{1}{2\sigma_b}\exp\left(-\frac{|x-\mu_b|}{\sigma_b}\right) \tag{4.4}$$

If $\mu_o = 60$, $\mu_b = 40$, $\sigma_o = 10$ and $\sigma_b = 5$, find the threshold that minimises the fraction of misclassified pixels, when we know that the object occupies two thirds of the full image.

We have $\theta = 2/3$. Substitute in (4.2):

$$\begin{aligned}
\theta\frac{1}{2\sigma_o}\exp\left(-\frac{|x-\mu_o|}{\sigma_o}\right) &= (1-\theta)\frac{1}{2\sigma_b}\exp\left(-\frac{|x-\mu_b|}{\sigma_b}\right) \\
\exp\left(-\frac{|x-\mu_o|}{\sigma_o}+\frac{|x-\mu_b|}{\sigma_b}\right) &= \frac{\sigma_o(1-\theta)}{\sigma_b\theta} \\
-\frac{|x-\mu_o|}{\sigma_o}+\frac{|x-\mu_b|}{\sigma_b} &= \ln\frac{10\times\frac{1}{3}}{5\times\frac{2}{3}} = \ln 1 = 0
\end{aligned}$$

Consider the following cases:
$t < \mu_b < \mu_o$: It leads to $t_1 = 20$
$\mu_b < t < \mu_o$: It leads to $t_2 = 47$
$\mu_b < \mu_o < t$: It leads to $t = 20$ which contradicts the assumption that $\mu_o < t$ and so it is rejected.

So, only pixels with grey value between 20 and 47 should be classified as background to minimise the error.

[60]