

$$1. a) i) P(S_1) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}. \quad [7.20]$$

$$(ii) S_1 \cup S_2 = \{a,b,c\}$$

$$(iii) S_1 \cap S_2 = \{a,b\}$$

$$(iv) S_2 - S_1 = \{ \emptyset \}$$

$$(v) |S_1 \cup S_2| = 3 \quad [6 \text{ MARKS}]$$

b) Finite:  $S_1$  from (a)

Infinite - Countable:  $\mathbb{N}$

Infinite - Uncountable:  $\mathbb{R}$

[3 MARKS]

c) (i) No, e.g.  $(2,1) \in R$  but  $(1,2) \notin R$ .

(ii) No, e.g.  $(2,1) \in R$  and  $(4,2) \in R$  but

$(4,1) \notin R$ .

(iii) No, e.g.  $(1,1) \notin R$ .

(iv) No, e.g. there is no element  $(1,x)$  for any  $x$ .

(v) No - (ii) implies this. [6 MARKS]

d) (i)  $\forall x (J(x) \rightarrow L(x))$

(ii)  $\exists x (L(x) \wedge \neg J(x))$

(iii)  $\forall x (J(x) \rightarrow A(x, \text{Jones}))$

(iv)  $\forall x \forall y (A(x,y) \wedge J(x) \rightarrow J(y))$ . [5 MARKS]

e) Simplify (i): b

Modus ~~Tollens~~ w/ (ii):  $\neg c$

[2 MARKS]

1. f) (i) From the theorem,  $f(x)$  is  $O(x^3)$ .

$$|f(x)| = |x^3 + 2x^2 + 1| \stackrel{\approx}{\approx} |x^3| + |2x^2| + 1 \quad \text{for } x > 0$$

$$\geq |x^3|$$

So with any  $\kappa$ , e.g.  $\kappa = 1$   $\Delta$  with  $c = 1$ ,

$$\forall x ((x > \kappa) \rightarrow |f(x)| \geq c|x^3|) \quad \square$$

(ii) `proc1 (int x) {`  
     for  $i = 1$  to  $x * x * x + 2 * x * x + 1 - 4$   
          $avar = avar * 2$ ;  
`}`

(iii) `proc2 (int a) {`  
     if  $a = 1$   
         return  $2 * x * x$ ;  
     else  
         return  $proc2(x-1) * proc2(x-1)$ ; [9 MARKS]  
`}`

g) let  $a > 1$  be a real number,  $b > 1$  be an integer,  
 $c > 0$  be a real number and  $d > 0$  be a real  
 number

let  $f$  be an increasing function s.t.

$$f(n) = a f(n/b) + cn^d \quad \text{whenever } n = b^k$$

for positive integer  $k$ .

- (i) If  $a < b^d$ ,  $f(n)$  is  $O(n^d)$   
 (ii) If  $a = b^d$ ,  $f(n)$  is  $O(n^d \log n)$   
 (iii) If  $a > b^d$ ,  $f(n)$  is  $O(n^{\log_b a})$ . [9 MARKS]

$$2. a) \text{ratapprox}(0, \sqrt{2}) = (1, 1)$$

$$\text{ratapprox}(1, \sqrt{2}) = (3, 2)$$

$$\text{ratapprox}(2, \sqrt{2}) = (7, 5) \quad [6 \text{ MARKS}]$$

$$b) a_n = 1 + a_{n-1}, \quad n \geq 1$$

$$a_n = n$$

This is  $O(n)$ , as are all other factors, so  
exec time is  $O(n)$ . [6 MARKS]

$$c) \forall x (\neg R(x) \rightarrow \forall y (R(y) \rightarrow (y < x) \vee (y > x)))$$

This is true. [6 MARKS]

$$d) r_d = \{p \mid p \in \mathbb{Q} \wedge p <_Q r\}$$

$$(i) \quad r_d \subset \mathbb{Q}$$

$r_d \subseteq \mathbb{Q}$  from defn

$r_d \neq \mathbb{Q}$  as, for example,  $r+1 \in \mathbb{Q}$  but  
 $r+1 \not<_Q r \Rightarrow (r+1) \notin r_d$ .

$$(ii) \quad r_d \neq \emptyset \text{ as, for example, } (r-1) \in \mathbb{Q}$$

Also  $(r-1) <_Q r \Rightarrow (r-1) \in r_d$ .

$$(iii) \quad q \in S \Rightarrow q \in \mathbb{Q} \wedge q <_Q r$$

$$p <_Q q \Rightarrow p <_Q q <_Q r$$

$$\Rightarrow p <_Q r$$

Also  $p \in \mathbb{Q}$ . Thus  $p \in S$ .

◻

2. (d) (iv) Take  $q = \frac{1}{2}(p+r)$ .

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Then  $q \in Q$  and  $p <_Q q <_Q r$

$\Rightarrow p <_Q q$ . [6 MARKS]

e) (i) For any set  $x$ ,  $x \subseteq x \Rightarrow x \leq_d x$ .

(ii)  $(x \leq_d y) \wedge (y \leq_d x)$

$\Rightarrow (x \subseteq y) \wedge (y \subseteq x)$

$\equiv x = y$  (by definition of set equality)

(iii)  $(x \leq_d y) \wedge (y \leq_d z)$

$\Rightarrow (x \subseteq y) \wedge (y \subseteq z)$

$\Rightarrow x \subseteq z$

$\Rightarrow x \leq_d z$ .

(iv) We wish to show  $(x \leq_d y) \vee (y \leq_d x)$

i.e.  $(x \subseteq y) \vee (y \subseteq x)$ .

If  $x = y$ , this is true.

Otherwise  $\exists q \in Q$  s.t.  $q \in x$  but  $q \notin y$  (\*)  
or  $q \in y$  but  $q \notin x$ .

Take (\*), w.l.o.g.

Then  $\forall p \in y$  ( $p <_Q q$ ) (from behind defn).

$\Rightarrow \forall p \in y$  ( $p \in x$ ) i.e.  $y \subseteq x$ .

[6 MARKS]

2. f) ~~g)~~ ~~h)~~  $T = \{p \mid p \in \mathbb{Q} \wedge (p^2 \leq_{\mathbb{Q}} 2) \vee (p < 0)\}$ .

(i)  $T \subseteq \mathbb{Q}$  by defn.

$T \neq \mathbb{Q}$  as, for example,  $2 \in \mathbb{Q}$  but  $2^2 \not\leq_{\mathbb{Q}} 2$   
 $\Rightarrow 2 \notin T$ .

(ii) ~~g)~~  $T \neq \emptyset$  as, for example,  $1 \in T$  since  $1 \in \mathbb{Q}$   
 &  $1^2 = 1 \leq_{\mathbb{Q}} 2$ .

(iii)  $q \in T \Rightarrow q \in \mathbb{Q}$   
 and  $q^2 \leq_{\mathbb{Q}} 2$  or  $q <_{\mathbb{Q}} 0$ .

Let  $q^2 <_{\mathbb{Q}} 0$ . Then  $p <_{\mathbb{Q}} q$

$\Rightarrow p <_{\mathbb{Q}} q <_{\mathbb{Q}} 0 \Rightarrow p <_{\mathbb{Q}} 0$

$\Rightarrow p \in T$ .

Let  $0 \leq_{\mathbb{Q}} q^2 <_{\mathbb{Q}} 2$ . Then  $p <_{\mathbb{Q}} q$

~~$p <_{\mathbb{Q}} 0$~~   $\Rightarrow p \in T$ , or  $p = 0 \Rightarrow p \in T$

or  $0 <_{\mathbb{Q}} p$ .

Now  $p^2 <_{\mathbb{Q}} pq$  (since  $0 <_{\mathbb{Q}} p$ )

$<_{\mathbb{Q}} q^2$  (since  $p <_{\mathbb{Q}} q$ )

$<_{\mathbb{Q}} 2$  (since  $q^2 <_{\mathbb{Q}} 2$ ).

$\Rightarrow p \in T$ .

2. (iv) Choose  $q = p + \frac{1}{n}$  with  $n$  as defined in "hint".

Then  $q$  is rational, since  $p$  is rational.

$$\text{Now } q^2 = p^2 + \frac{1}{n} (2p + \frac{1}{n}) <_Q p^2 + \frac{1}{n} (2p + 1)$$

as  $n >_Q 1$ .

$$n >_Q \frac{2p+1}{2-p^2} \Rightarrow \frac{1}{n} <_Q \frac{2-p^2}{2p+1}$$

$$\Rightarrow q^2 <_Q p^2 + 2 - p^2$$

$$= 2 \quad (*)$$

$$\text{Also } n >_Q 1 \Rightarrow q = p + \frac{1}{n} >_Q p \quad (+)$$

$$\text{So (i) } q \in S \text{ for } (*)$$

$$\text{(ii) } p <_Q q \text{ for } (+) \quad \square$$

~~As a result,  $q$~~

[10 MARKS]

3. a) (i) Let  $x \in B = \mathbb{R}$ .

Then  $2^x \in \mathbb{R}$  and  $2^x > 0 \Rightarrow 2^x \in \mathbb{R}_+$ .

So  $f(2^x, 0) = x \quad \square$ . [3 MARKS]

(ii) Choose  $A = \mathbb{R}_+ \times \{0\}$

Proof above holds for surjectivity.

For injectivity,

$$f(x_1, y_1) = f(x_2, y_2)$$

But  $y_1 = y_2 = 0$ .

$$\text{Also } \log_2(x_1 + 0) = \log_2(x_2 + 0)$$

$$\Rightarrow x_1 = x_2 \quad \square. \quad [6 \text{ MARKS}]$$

(iii) It is possible to obtain any ~~positive~~ non-negative value  $v \in \mathbb{R}_+ \cup \{0\}$  from

$$f(2^v, 0) = v \quad \& \quad 2^v \geq 1$$

~~Also zero is possible iff~~

Negative values are not possible - we

would require  $f(x, y) < 0$

$$\Rightarrow \log_2(x+y) < 0$$

So either  $\log_2 x < 0 \Rightarrow x < 1$  or

or  $\log_2(x+1) < 0 \Rightarrow x+1 < 1$

i.e.  $x < 0$ .

So image is  $\mathbb{R}_+ \cup \{0\}$

[6 MARKS]

3. (b)(i)  $R^n \subseteq R \Rightarrow R$  is transitive:

$R^n \subseteq R \Rightarrow R^2 \subseteq R$ .  $(a,b) \in R$  and  $(b,c) \in R$   
 then  $(a,c) \in R^2$ . But  $R^2 \subseteq R \Rightarrow (a,c) \in R$ .  
 So  $R$  is transitive.

$R$  is transitive  $\Rightarrow R^n \subseteq R$

True for  $n=1$ . Use induction to show  $R^{n+1} \subseteq R$   
 assuming  $R^n \subseteq R$ .

Consider  $(a,b) \in R^{n+1} = R \cdot R^n$

$\Rightarrow \exists x ((a,x) \in R \wedge (x,b) \in R^n)$

$R^n \subseteq R \Rightarrow (x,b) \in R$ .  $R$  is transitive

$\Rightarrow (a,b) \in R$ . So  $R^{n+1} \subseteq R$ . [6 MARKS]

(ii)  $\forall a \in A \exists b \in B ((a,b) \in R)$

$\wedge \forall a \in A \forall b \in B \forall c \in B ((a,b) \in R \wedge (a,c) \in R$

$\rightarrow b=c)$ . [6 MARKS]

(iii) Trivially,  $f: \{0\} \rightarrow \{0\}$   
 $f(0)=0$  is transitive.

[3 MARKS]

4. a) (i)  $T(\text{Steven}) \wedge I(\text{Steven})$

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(ii)  $G(\text{Steven}) \wedge \forall x (I(x) \rightarrow G(x))$

(iii)  $\exists x (T(x) \wedge I(x))$

$\wedge \forall x \forall y (T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x=y)$

(iv)  $\forall x (T(x) \wedge I(x) \rightarrow x = \text{Steven})$

(v)  $T(\text{Amanda}) \wedge \neg T(\text{James})$

(vi)  $\neg I(\text{Amanda})$

[12 MARKS]

b) Simplify (iii)  $\Rightarrow \forall x \forall y (T(x) \wedge I(x) \wedge T(y) \wedge I(y) \rightarrow x=y)$

Universal Instantiation

$\forall x (T(x) \wedge I(x) \wedge T(\text{Steven}) \wedge I(\text{Steven}) \rightarrow x = \text{Steven})$

Hypothesis  $\Rightarrow \forall x (T(x) \wedge I(x) \rightarrow x = \text{Steven})$

[9 MARKS]

c) Universal Instantiation on (iv)

$T(\text{Amanda}) \wedge I(\text{Amanda}) \rightarrow \text{Amanda} = \text{Steven}$

Modus Ponens

$\neg (T(\text{Amanda}) \wedge I(\text{Amanda}))$

$\equiv \neg T(\text{Amanda}) \vee \neg I(\text{Amanda})$  (\*)

Simplify (v)  $T(\text{Amanda})$  (+)

Disjunctive Syllogism (\*) & (+)

$\Rightarrow \neg I(\text{Amanda})$

[9 MARKS]

5. a)  $f(x)$  is  $O(g(x)) \equiv \exists c \in \mathbb{R}^+ \exists \kappa \in \mathbb{R}^+ \forall x (x > \kappa) \rightarrow (|f(x)| \leq c |g(x)|)$   
 $f(x)$  is  $\Omega(g(x)) \equiv \exists c \in \mathbb{R}^+ \exists \kappa \in \mathbb{R}^+ \forall x (x > \kappa) \rightarrow (|f(x)| \geq c |g(x)|)$   
 $f(x)$  is  $\Theta(g(x)) \equiv [f(x) \text{ is } O(g(x))] \wedge [f(x) \text{ is } \Omega(g(x))]$ .

[6 MARKS]

- b)  $\exists \kappa_1, \kappa_2, c_1, c_2$  s.t.

$$|f_1(x)| \leq c_1 |g_1(x)|, \quad x > \kappa_1$$

$$\& |f_2(x)| \leq c_2 |g_2(x)|, \quad x > \kappa_2$$

By the triangle inequality,

$$|f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)|$$

$$\leq c_1 |g_1(x)| + c_2 |g_2(x)|, \quad x > \max(\kappa_1, \kappa_2)$$

$$\leq c_1 \max(|g_1(x)|, |g_2(x)|) +$$

$$c_2 \max(|g_1(x)|, |g_2(x)|)$$

$$= (c_1 + c_2) \max(|g_1(x)|, |g_2(x)|)$$

So with  $c = c_1 + c_2$ ,  $\kappa = \max(\kappa_1, \kappa_2)$ ,

$$f_1(x) + f_2(x) \text{ is } O(\max(|g_1(x)|, |g_2(x)|)).$$

[6 MARKS]

- c) 

```
proc1(int n) {
  total := 1;
  for i = 1 to 2*n
    total := total * n;
}
```

(Assuming loop  
term. cost  
evaluated once)

[6 MARKS]

- d) 

```
proc2(int n) {
  if n = 1
    return 3*n;
  else if n = 2
    return 3*n*n*n;
}
```

else  
return  $n * \text{proc2}(n \text{ div } 3)$ ;  
}

[6 MARKS]

5. (e) Proc 1 has  $O(n)$  exec time (#mults)

Proc 2 has  $O(\log n)$  exec time (#mults)

Proc therefore has  $O(\max(n, \log n))$

$$= O(n)$$

$k=1$

[6 MARKS]