IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2008

EEE PART III/IV: MEng, BEng and ACGI

DISCRETE MATHEMATICS AND COMPUTATIONAL COMPLEXITY

Tuesday, 27 May 2:00 pm

Time allowed: 3:00 hours

Corrected Copy

Q 3

There are FIVE questions on this paper.

Answer Question One (40 marks), Question Two (40 marks), and TWO of Questions Three to Five (30 marks each). Note that this paper is marked out of 140.

Any special instructions for invigilators and information for candidates are on page 1.

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NOTATION

The following notation may be used throughout this paper:

$\mathbb{R}$: The set of real numbers.

$\mathbb{R}_+$: The set of positive real numbers.

$\mathbb{Z}$: The set of integers.

$\mathbb{Z}_+$: The set of positive integers.

$\mathbb{N}$: The set of natural numbers.

$\mathbb{Q}$: The set of rational numbers.

$\mathscr{P}(S)$: The power set of set S.

The Questions

1. **[Compulsory]**

a) For the sets $S_1 = \{a, b, c\}$ and $S_2 = \{a, b\}$,

i) list all the subsets of S_1,

ii) list the elements of $S_2 \cup S_1$,

iii) list the elements of $S_2 \cap S_1$,

iv) list the elements of $S_2 - S_1$,

v) state the cardinality of $S_2 \cup S_1$.

[6]

b) Identify one finite set, one countably infinite set, and one uncountably infinite set.

[3]

c) Consider the relation $R = \{(a,b) | a = 2b\}$ on the set $\mathbb{Z}$. Answer each of the following questions about this relation, supplying an appropriate justification for your answer in each case.

i) Is R symmetric?

ii) Is R transitive?

iii) Is R reflexive?

iv) Is R a function?

v) Is $R \cdot R$ a subset of R?

[6]

d) Express each of the following English statements using predicate logic. You should take the universe of discourse as the set of all people, and use the following predicates: $J(x)$ denotes the statement 'x is a judge', $L(x)$ denotes the statement 'x is a lawyer', $A(x,y)$ denotes the statement 'x admires y'.

i) All judges are lawyers.

ii) Not all lawyers are judges.

iii) All judges admire Judge Jones.

iv) All judges admire only judges.

[5]

e) Given the two hypotheses (i) $a \wedge b$ and (ii) $c \rightarrow \neg b$, apply standard rules of inference to conclude $\neg c$. State the rule of inference used at each step in your working.

[2]

f) Let $f(x) = x^3 + 2x^2 + 1$ and $g(x) = x^3$.

i) Show that $f(x)$ is $\Theta(g(x))$. (You may use the theorem for big-O of polynomials proved in the lecture notes).

ii) Write some pseudo code for a procedure `proc1`(x) that executes exactly $f(x)$ multiplications when called with parameter $x \geq 1$.

iii) Write some pseudo code for a procedure `proc2`(x) that executes 2^x multiplications when called with parameter $x \geq 1$.

[9]

g) State the Master Theorem.

[9]

2. **[Compulsory]**

Fig. 2.1 shows some code that finds rational numbers approximately equal to $\sqrt{2}$ when called with its real argument equal to $\sqrt{2}$. It returns a pair (a, b) of integers, such that $a/b \approx \sqrt{2}$; the larger the value of n, the better the approximation. All operations in the code are performed in real arithmetic, and `round` rounds its argument to the nearest integer (away from zero in the case of a tie).

```
ratapprox( integer n, real r )
begin
  if n = 0 then
    return (round(r),1)
  else
    intpart := round(r)
    rest := r - intpart
    reciprocal := 1/rest
    (num,den) := ratapprox( n − 1, reciprocal )
    a := intpart * num + den
    b := num
    result := (a,b)
  end
end
```

Figure 2.1 Pseudo-code for approximating a real number.

a) Evaluate `ratapprox`$(0, \sqrt{2})$, `ratapprox`$(1, \sqrt{2})$, and `ratapprox`$(2, \sqrt{2})$.

[6]

b) Write a recurrence relation for the number of times `ratapprox`(n,r) performs a division operation, and hence or otherwise provide a big-O expression for the run-time of `ratapprox` in terms of n.

[4]

c) $\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} \notin \mathbb{Q}$. Let $R(x)$ be the predicate 'x is rational', defined over the set of real numbers. Using $\mathbb{R}$ as the universe of discourse, express, using symbolic logic, the statement 'no matter which irrational real number x I choose, every rational number is either less than or greater than x'. State the truth value of this proposition.

[6]

In the remainder of this question, the relation $<_{\mathbb{Q}}$ is the relation 'less than' on the set of rational numbers, and the operation $+_{\mathbb{Q}}$ is the addition operation on the set of rational numbers.

The previous part of this question suggests a way of *defining* real numbers in terms of rational numbers. A formal definition of a real number is a Dedekind set. A Dedekind set is a set S having the following properties:

- $S \subset \mathbb{Q}$,
- $S \neq \emptyset$,
- $\forall q \in \mathbb{Q} \forall p \in \mathbb{Q}((q \in S) \wedge (p <_{\mathbb{Q}} q) \rightarrow (p \in S))$,
- $\forall p \in S \exists q \in S (p <_{\mathbb{Q}} q)$.

The set $\mathbb{R}_d$ is defined as the set of all Dedekind sets. Under this definition, the Dedekind set corresponding to a rational number $r \in \mathbb{Q} \subseteq \mathbb{R}$ is given by $r_d = \{p | p \in \mathbb{Q} \wedge p <_{\mathbb{Q}} r\}$.

d) Show that r_d is indeed a Dedekind set for all rational numbers r, *i.e.* $\forall r \in \mathbb{Q} r_d \in \mathbb{R}_d$.

[6]

e) We can define the relation $\leq_d = \{(x,y) | x \subseteq y\}$ on the set of Dedekind sets.

Show that each of the following properties holds, where the universe of discourse is $\mathbb{R}_d$.

i) $\forall x (x \leq_d x)$,

ii) $\forall x \forall y ((x \leq_d y) \wedge (y \leq_d x) \rightarrow (x = y))$,

iii) $\forall x \forall y \forall z ((x \leq_d y) \wedge (y \leq_d z) \rightarrow (x \leq_d z))$,

iv) $\forall x \forall y ((x \leq_d y) \vee (y \leq_d x))$.

[8]

f) A set corresponding to $\sqrt{2}$ could be defined in terms of rational numbers as $T = \{p | p \in \mathbb{Q} \wedge ((p^2 <_{\mathbb{Q}} 2) \vee p <_{\mathbb{Q}} 0)\}$. Show that this is a Dedekind set. (*Hint:* For the final property of Dedekind sets, consider selecting an integer n satisfying $n >_{\mathbb{Q}} \frac{2p+1}{2-p^2}$ and $n >_{\mathbb{Q}} 1$. Show that $(p + 1/n)^2 = p^2 + \frac{1}{n}\left(2p + \frac{1}{n}\right) <_{\mathbb{Q}} 2$, and draw the appropriate conclusions.)

[10]

3. a) Consider the function $f : A \to B$ defined by ~~$f(x)$~~ $f(x,y)$ $= \log_2(x+y)$.

i) For $A = \mathbb{R}_+ \times \mathbb{R}_+$ and $B = \mathbb{R}$, show that f is a surjection.

[3]

ii) Show that for $B = \mathbb{R}$, f is a bijection for a suitable choice of A.

[6]

iii) Find the image of the set $\{(x,y) | x \in \mathbb{R}_+ \wedge y \in \{0,1\} \wedge x \geq 1\}$ under the function f. Justify your answer.

[6]

b) i) Prove that a relation R on a set is transitive iff $\forall n \in \mathbb{Z}_+ (R^n \subseteq R)$.

[6]

ii) For a relation $R \subseteq A \times B$, use symbolic logic to express a proposition that is true iff R is a function.

[6]

iii) Give an example of function that is also a transitive relation.

[3]

4. For this question, you should use the set of all people as the universe of discourse, along with the following predicates: $T(x)$ denotes the statement 'x is tall', $I(x)$ denotes the statement 'x is an Imperial College student', $G(x)$ denotes the statement 'x is a geek'.

a) Express the following sentences as propositions using appropriate symbolic logic.

i) 'Steven is a tall Imperial College student'.

ii) 'Like all Imperial College students, Steven is a geek'.

iii) 'There is exactly one tall student at Imperial College'.

iv) 'There are no tall Imperial College students except Steven'.

v) 'Amanda is tall, whereas James is short'.

vi) 'Amanda is not an Imperial College student'.

[12]

b) Using your answers to Part (a)(i) and Part (a)(iii) as your axioms, formally prove your answer to Part (a)(iv), stating the rule of inference used at each step in your argument.

[9]

c) Using your answers to Part (a)(iv) and Part (a)(v) as your axioms, formally prove your answer to Part (a)(vi), stating the rule of inference used at each step in your argument.

[9]

5. a) Define what is meant by the statements $f(x)$ is $O(g(x))$, $f(x)$ is $\Omega(g(x))$, and $f(x)$ is $\Theta(g(x))$, using appropriate symbolic logic.

[6]

b) Prove that if $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then $f_1(x) + f_2(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.

[6]

c) Construct some code for a procedure `proc1(integer n)` that executes $2n+1$ multiplication operations when called with parameter $n > 0$.

[6]

d) Further construct some code for a procedure `proc2(integer n)` that executes $f(n)$ multiplication operations when called with parameter $n > 0$, where $f(1) = 1$, $f(2) = 2$, and $f(n) = f(\lfloor n/3 \rfloor) + 1$ for $n > 2$.

[6]

e) What is the lowest value of k such that the number of multiplications executed by `proc(integer n)`, shown in Fig. 5.1, is $O(n^k)$. Justify your answer.

[6]

```
proc( integer n )
begin
    proc1(n)
    proc2(n)
end
```

Figure 5.1 Pseudo-code for procedure `proc`. Procedures `proc1` and `proc2` behave as explained in parts (c) and (d), respectively.