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June 08

SOLUTIONS (E2.6, Control Engineering, 2008)

1. (a) (i) Applying Newton's laws on the mass,

$$u(t) = M\ddot{y}(t) + D\dot{y}(t) + Ky(t).$$

- (ii) Taking Laplace transforms,

$$\frac{y(s)}{u(s)} = \frac{1}{Ms^2 + Ds + K},$$

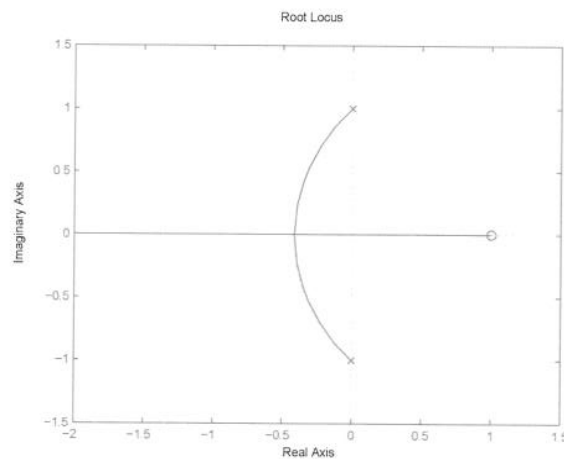
- (iii) The response is critically damped when the two poles are equal. Thus $D = 2$.

- (iv) When $u(s) = 1/s$, we have

$$y(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1}$$

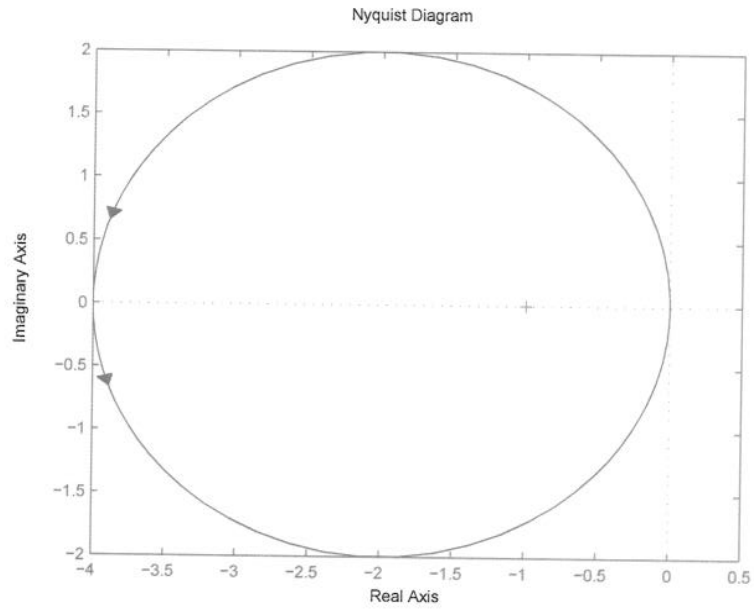
and so $y(t) = 0.5 + 0.5e^{-2t} - e^{-t}$.

- (b) (i) The root locus is shown below. The breakaway point is at the negative root of $dG(s)/ds$ and is given by $\sigma_b = 1 - \sqrt{2} = -0.4142$. The angle of departure is given by the angle criterion: $135^\circ - (\theta_d + 90^\circ) = 180^\circ$ and so $\theta_d = -135^\circ$.



- (ii) The Routh array gives the stability range as $0 < K < 1$.
- (iii) For critical damping, the closed loop poles must be placed at the breakaway point σ_b . The corresponding gain is obtained from the gain criterion: $k = -G(s)^{-1}|_{s=\sigma_b} = 2(\sqrt{2} - 1) = 0.8284$.

(c) (i) The Nyquist diagram is shown below:



- (ii) From the Nyquist theorem, $N = Z - P$ where N ($= -1$ in this case) is the number of clockwise encirclement of the point -1 , P ($= 1$ in this case) is the number of open loop unstable poles, and Z is the number of closed-loop unstable poles. Hence $Z = 0$ and the closed-loop is stable.
- (iii) Since the negative real-axis intercept is at -4 , the gain margin is 0.25 .

2. (a) For an op-amp in the negative feedback mode, the transfer function is $-Z_f/Z_i$ where Z_f is the feedback impedance and Z_i is the input impedance. Thus the transfer function between $V_r(s)$ and $V(s)$ is

$$-\frac{s + 1/C1}{s}$$

and that between $V_o(s)$ and $V(s)$ is the same.

- (b) The transfer function between $V(s)$ and $V_o(s)$ can be obtained in two stages. Let V_1 be the voltage at the output of O_2 . Then $V_1(s) = -\frac{1}{s+1}V(s)$. Also, $V_o(s) = -\frac{1}{s+2}V_1(s)$. It follows that

$$V_o(s) = \frac{1}{(s + 1)(s + 2)}V(s).$$

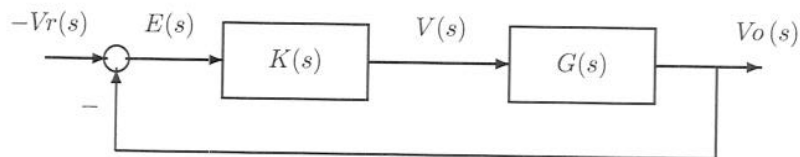
- (c) The block diagram is given below, where

$$G(s) = \frac{1}{(s + 1)(s + 2)}$$

and

$$K(s) = \frac{s + 1/C1}{s}.$$

$K(s)$ is a proportional-plus-integral (PI) type compensator.



- (d) Now,

$$E(s) = \frac{-V_r(s)}{1 + G(s)K(s)}.$$

Since $V_r(s) = 1/s$, it follows from the final value theorem that

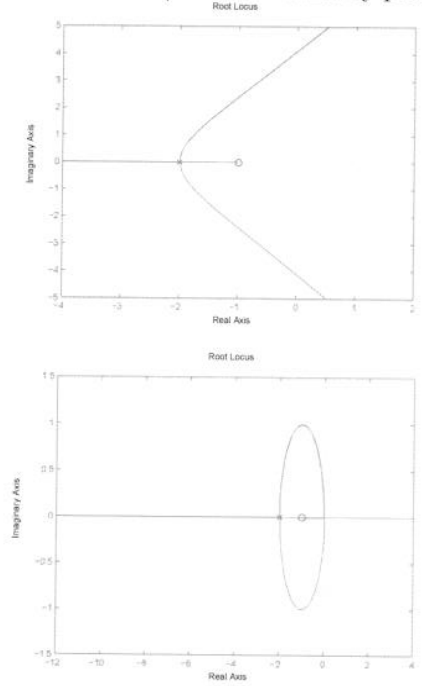
$$E_{ss}^{step} = \lim_{s \rightarrow 0} sE(s) = 0.$$

- (e) Here $V_r(s) = 1/s^2$. It follows from the final value theorem that

$$|E_{ss}^{ramp}| = \left| \lim_{s \rightarrow 0} sE(s) \right| = 2C1.$$

Thus, for $|E_{ss}^{ramp}| \leq \epsilon$, it follows that we require $C1 \leq \epsilon/2$, which is the maximum value.

3. (a) The plot is shown below. The angles of the asymptotes are $\pm 60^\circ, 180^\circ$ and the centre is at $-7/3$. The breakaway point is at -2 .



- (b) The characteristic equation is $s^4 + 8s^3 + 24s^2 + (32+k)s + 16+k = 0$.

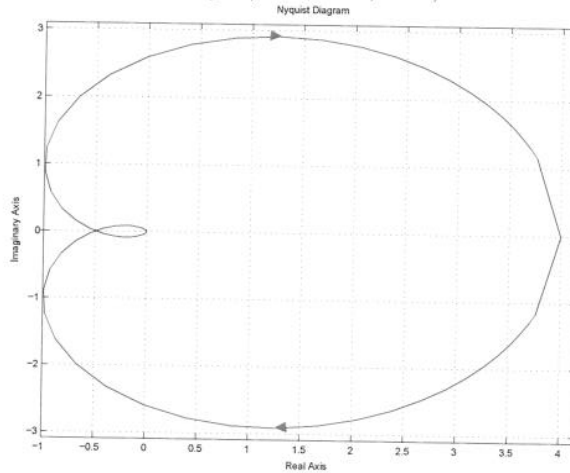
The Routh array :

| | | | |
|-------|------------------------------|----------|--------|
| s^4 | 1 | 24 | 16 + k |
| s^3 | 8 | $32 + k$ | |
| s^2 | $\frac{160-k}{8}$ | $16 + k$ | |
| s | $\frac{4096+64k-k^2}{160-k}$ | | |
| 1 | $16 + k$ | | |

The positive root of $4096 + 64k - k^2 = 0$ is $\bar{k} = 103.5542$. Since we require no sign changes in the first column, $0 < k < \bar{k}$.

- (c) We have marginal stability when $k = \bar{k}$. Substituting in the array, the auxiliary polynomial is $\frac{160-\bar{k}}{8}s^2 + 16 + \bar{k} = 0$ which has roots at $\pm j\bar{\omega}$, $\bar{\omega} = 16.9443$ rad/s. So the frequency of oscillations is $\bar{\omega}$.
- (d) The compensator is $K(s) = k(s+z)$. To satisfy the specifications we place CL poles at $s_0 = -1 \pm j$. An inspection of the root locus shows that we must use negative k . Let the angle between s_0 and z be θ and that between s_0 and -1 be θ_0 . Then $\theta_0 = 180^\circ - \tan^{-1}(2)$. The angle criterion gives $\theta + \theta_0 - (4 \times 90^\circ) = 0^\circ$ or $\theta = \tan^{-1}(2) - 180^\circ$. Thus $z = 1$. The root locus is shown above. The gain criterion gives $k = -(s+2)^4 / (s+1)^2 |_{s=-2+j2} = -4$ so $K(s) = -4(s+1)$.

4. (a) The Nyquist plot is shown below. The real-axis intercepts can be found by evaluating the Routh array. This gives intercepts at $\omega_i = 0, \pm\sqrt{3}, \infty$ and so $G(j\omega_i) = 4, -0.5, -0.5, 0$.



- (b) (i) The number of unstable closed-loop poles is determined by the number of encirclements by $G(s)$ of the point -1 , which is zero. Thus the closed-loop is stable since $G(s)$ has no unstable poles.
- (ii) Since the real-axis intercept is at -0.5 , the gain margin is 2. The intercept with the unit circle centred on the origin occurs when $4 = \sqrt{(\omega^2 + 1)^3}$ or $\omega_p = \sqrt{16^{\frac{1}{3}} - 1} = 1.2328$. Then $\angle G(\omega_p) \sim -153^\circ$ and the phase margin is $180^\circ + \angle G(\omega_p) \sim 27^\circ$.
- (c) (i) A PI compensator has the form $K(s) = K_p + K_i/s = K_p \frac{s + K_i/K_p}{s}$, with $K_p > 0$, $K_i > 0$. A PI compensator is a special form of phase-lag compensation and has high gain at low frequencies (infinite at DC) and close to K_p for high frequencies. The phase is close to -90° at low frequencies and tends to 0 at high frequencies.
- (ii) Thus PI compensation can increase low frequency gain and hence improve steady-state tracking since

$$|e(j\omega)| = \left| \frac{1}{1 + G(j\omega)K(j\omega)} \right| |r(j\omega)|$$

without increasing high frequency gain (thus degrading the gain margin). Care should be taken concerning the phase-margin since the phase lag may deteriorate this.

To reduce the destabilizing effect of the PI compensator we place the zero very near the pole, i.e. we choose $K_i \ll K_p$ so there is approximately a pole/zero cancellation at the origin.