IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2010

EEE/ISE PART II: MEng, BEng and ACGI

CONTROL ENGINEERING

Friday, 28 May 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : I.M. Jaimoukha, I.M. Jaimoukha

Second Marker(s) : S. Evangelou, S. Evangelou

1. a) Figure 1.1 illustrates an RLC circuit. The capacitor has capacitance C, the inductor has inductance L and the resistor resistance R. Take the input to be the applied voltage $v_i(t)$ and the output to be the voltage across the capacitor and resistor $v_o(t)$.

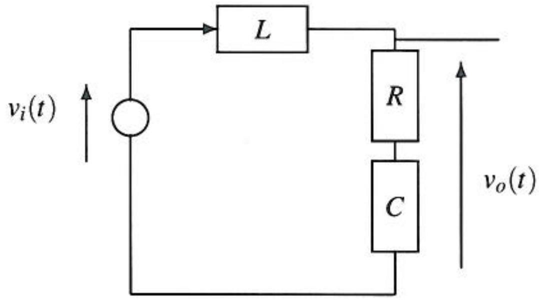

Figure 1.1

i) Determine $G(s)$, the transfer function relating v_o to v_i. [4]

ii) Let $v_i(t)$ be a unit step function applied at $t = 0$. Use the final value theorem, which should be stated, to find the steady-state value of $v_o(t)$. [5]

iii) Derive the value of R so that $G(s)$ is marginally stable. What is the frequency of oscillations? Give your answer in terms of L and C. [5]

b) In Figure 1.2 below, $G(s) = \frac{s+1}{s-1}$ and K is a variable gain.

i) Sketch the locus of the closed-loop poles for $0 \leq K < \infty$. [5]

ii) Using the gain criterion, find the value of K for which the closed-loop is marginally stable. [4]

iii) Find the range of $K \geq 0$ for which the closed-loop is stable. [4]

c) In Figure 1.2 below, $G(s) = \frac{1}{s-0.5}$ and K is a variable gain.

i) Draw the Nyquist diagram of $G(s)$ indicating real-axis intercepts. [5]

ii) Take $K = 0.25$. Use the Nyquist criterion, which should be stated, to determine the number of unstable closed-loop poles. [4]

iii) Take $K = 1$. Use the Nyquist criterion to show that the closed-loop is stable. Comment on the gain margin. [4]

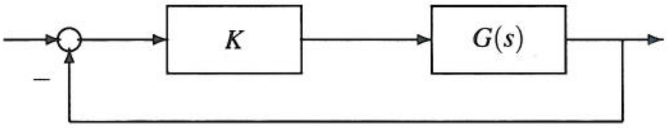

Figure 1.2

2. Figure 2 below shows the block diagram of a feedback system for voltage regulation, where $v_r(t)$ is the reference voltage, $v_o(t)$ is the supplied output voltage and R is a resistance which is fixed but has an unknown value.

The op-amp open-loop output voltage E is related to v_e as $E(s) = -G(s)v_e(s)$, where the transfer function $G(s)$:

- is second order,
- has no zeros,
- has a DC gain A,
- has two real stable poles with time constants 5 ms and 10 ms.

a) Derive an expression for $G(s)$ in terms of A. [7]

b) Derive an expression for $v_e(s)$ in terms of $v_r(s)$ and $v_o(s)$. [5]

c) Derive an expression for $v_o(s)$ in terms of $v_e(s)$. [5]

d) Hence, derive and draw a block diagram representation of the feedback loop. Assume the feedback gain to be equal to unity and that the reference and output signals are $-v_r(s)$ and $v_o(s)$, respectively. Indicate the signal $v_e(s)$ on the block diagram. [5]

e) Find A_{max}, the maximum value of the DC gain A such that the step response of the closed-loop system is non-oscillatory. Find also the corresponding closed-loop poles. [8]

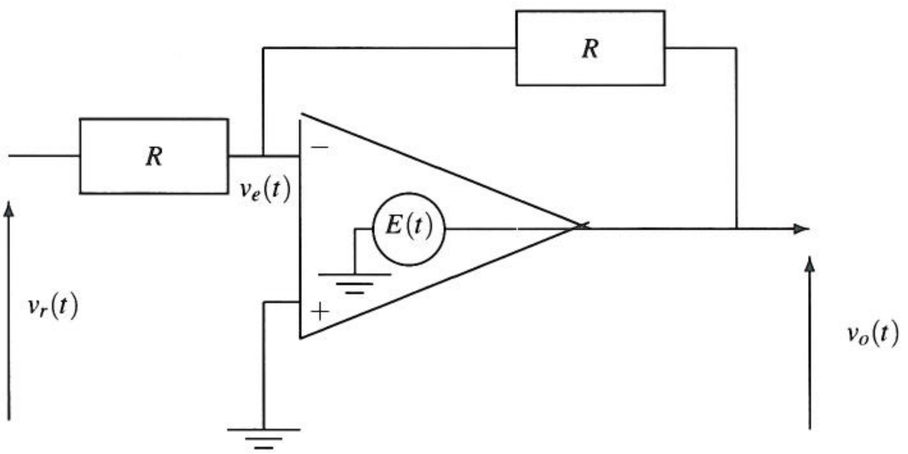

Figure 2

3. Consider the feedback system shown in Figure 3 below, where

$$G(s) = \frac{1}{s+1}.$$

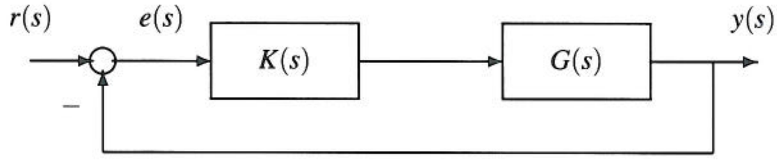

Figure 3

A feedback compensator $K(s)$ is required such that the following design specifications are satisfied:

- The closed-loop is stable.
- The closed-loop step response is non-oscillatory and has a settling time of 2 *s*.
- The DC gain of the transfer function from $e(s)$ to $y(s)$ is equal to 11.

a) Draw the root locus of $G(s)$ accurately for all $K \geq 0$. [5]

b) Derive the location of the closed-loop pole that satisfies the second design specification above. [5]

c) Show that the design specifications cannot be satisfied using a proportional compensator. [5]

d) Design a PD compensator that meets all the specifications. [5]

Hint: Define your compensator in terms of two parameters, say K_d and z. Next, obtain algebraic relations, perhaps involving the gain criterion, to satisfy the second and third specifications.

e) Draw the root locus of the compensated system. [5]

f) Evaluate the steady-state error of the closed-loop system for a unit step reference signal. [5]

4. Figure 4 below shows a feedback control system for which

$$G(s) = \frac{6}{(s+1)^3}$$

and $K(s)$ is the transfer function of a compensator.

a) Sketch the Nyquist diagram of $G(s)$, indicating the low- and high-frequency portions. Also, calculate the real-axis intercepts. [7]

b) Assume that $K = 1$. Show that the closed-loop is stable and determine the gain and phase margins. [7]

c) Without doing any actual design, briefly describe how a phase-lead compensator,

$$K(s) = \frac{1+s/\omega_0}{1+s/\omega_p}, \qquad 0 < \omega_0 < \omega_p,$$

would affect the gain and phase margins. Your answer should emphasize the difficulties involved in the design. [8]

d) Design a stabilising phase-lead compensator $K(s)$ such that the loop gain has the same DC gain as $G(s)$ and the gain margin of $G(s)K(s)$ is infinite. Draw a rough sketch of the Nyquist diagram of $G(s)K(s)$. [8]

Hint: You may consider using a special type of phase-lead compensator that implements a pole-zero cancellation.

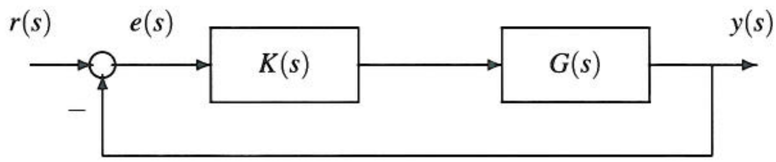

Figure 4

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1. a) i) Using the potential divider rule and the impedances we have

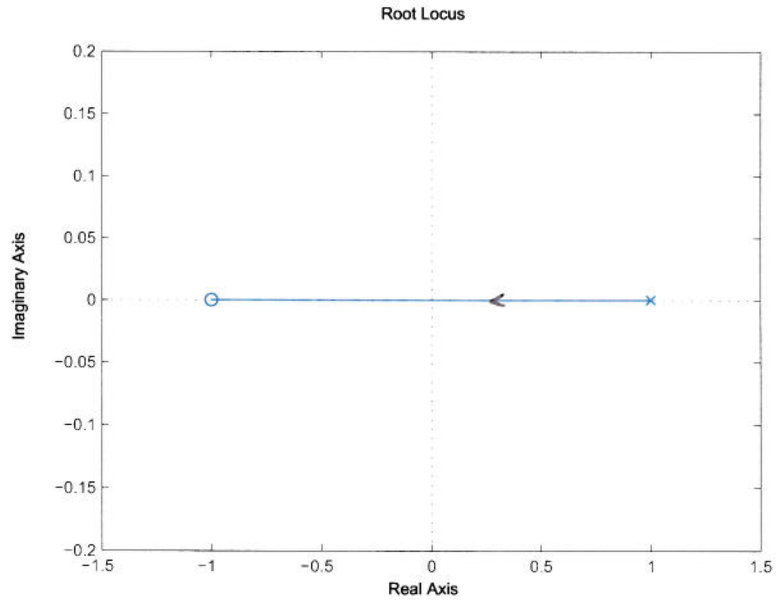
$$G(s) := \frac{v_o(s)}{v_i(s)} = \frac{sRC+1}{s^2LC+sRC+1}.$$

ii) Using the final value theorem and the fact that $v_i(s) = 1/s$,

$$\lim_{t\to\infty} v_o(t) = \lim_{s\to 0} s v_o(s) = \lim_{s\to 0} sG(s)v_i(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0) = 1.$$

iii) For marginal stability, the poles must be imaginary so $R = 0$. The frequency of oscillations isgiven by $\omega = \frac{1}{\sqrt{LC}}$.

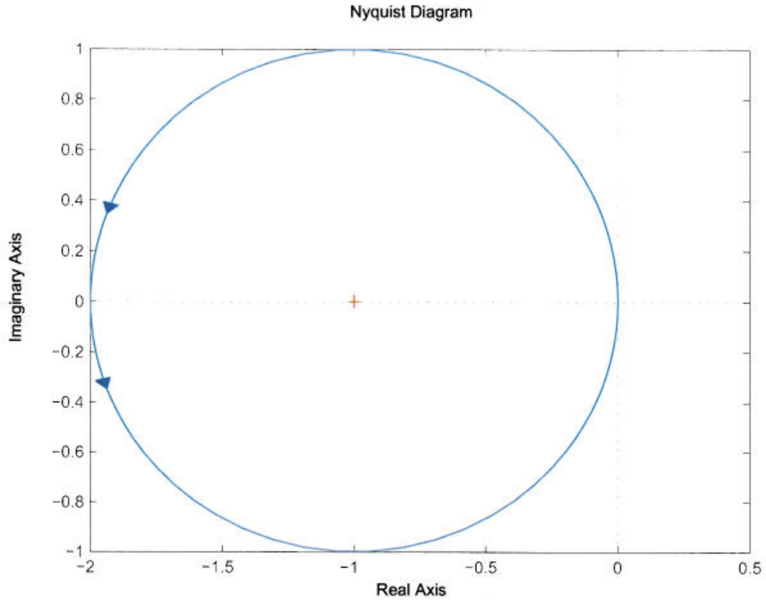
b) i) The root-locus is shown below.



ii) The closed-loop is marginally stable when at least one pole is on the imaginary axis and all others are in the left half-plane. It follows from the root-locus that the marginal pole pole is at $s = 0$. Using the gain criterion $K = -1/G(0) = 1$.

iii) It follows from the root-locus that the closed-loop is stable for all $K > 1$.

c) i) The Nyquist diagram is shown below. It is clear that the real axis intercepts are at -2 and 0.



ii) Let $K = 0.25$. The Nyquist criterion states that $N = Z - P$, where N is the number of clockwise encirclements by $G(s)$ of the point $-1/K$ as s traverses the Nyquist contour, which in this case is equal to 0; P is the number of unstable open-loop poles, which in this case is equal to 1; and Z is the number of unstable closed-loop poles. Thus there are $Z = N + P = 1$ unstable closed-loop poles.

iii) When $K = 1$, then $N = -1$, $P = 1$ and so $Z = N + P = 0$ and the closed-loop is stable. Since the gain can be increased without bound the system has infinite gain margin for increasing gain. The gain can also be decreased by 50% before losing stability.

2. a) A transfer function $G(s)$ with the required properties has the form

$$G(s) = \frac{A}{(1+sT_1)(1+sT_2)} = \frac{AT_1^{-1}T_2^{-1}}{(s+T_1^{-1})(s+T_2^{-1})} = \frac{2\times 10^4 \times A}{(s+10^2)(s+2\times 10^2)}.$$

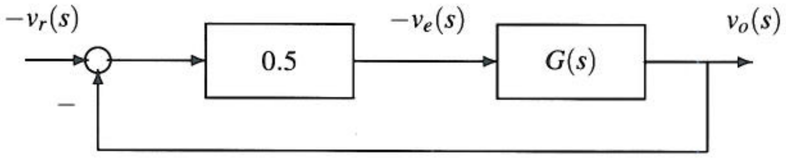
b) Using a potential divider rule at the op-amp input gives

$$\frac{v_e(s)-v_r(s)}{v_o(s)-v_r(s)} = \frac{R}{R+R} = \frac{1}{2} \Rightarrow -v_e(s) = -0.5v_r(s) - 0.5v_o(s) = 0.5\left((-v_r(s)) - v_o(s)\right).$$

c) At the op-amp output we have

$$E(s) = v_o(s) \Rightarrow v_o(s) = -G(s)v_e(s).$$

d) Using Parts (a) and (b), the block diagram becomes,



e) For non-oscillatory step response, the closed-loop is critically damped so that the closed-loop poles are real and equal. The characteristic equation is given by

$$1 + 0.5G(s) = 0$$

which can be written as

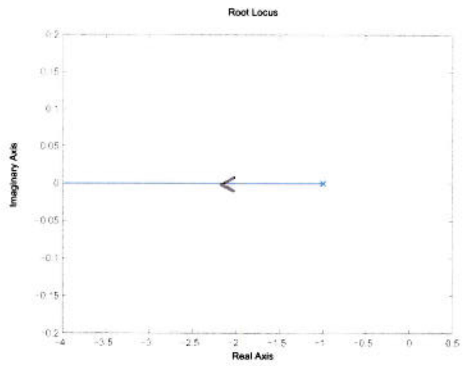
$$s^2 + 3\times 10^2 s + 2\times 10^4 + 10^4 \times A = 0$$

or

$$(s+1.5\times 10^2)^2 + (A-0.25)\times 10^4 = 0.$$

It follows that $A_{max} = 0.25$ and the corresponding two closed-loop poles are at -1.5×10^2.

3. a) The root-locus is shown below.

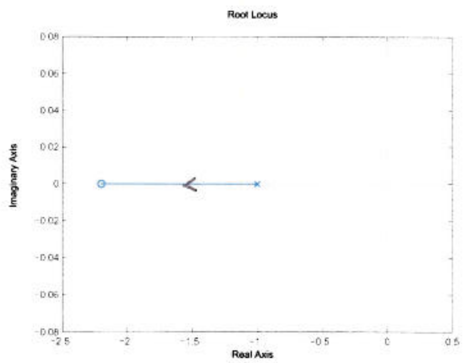


b) For a non-oscillatory response with a settling time of 2 seconds, the closed-loop pole must be located at -2.

c) Using the gain criterion, for a closed-loop pole at -2, $K = -1/G(-2) = 1$. The resulting DC gain is then equal to $KG(0) = 1$ and the third specification is not specified. Thus there does not exist a proportional compensator that satisfies the design specifications.

d) Following the hint, a PD compensator has the form $K(s) = K_d(s+z)$. To satisfy the second specification, the gain criterion requires that $1 - K_d(-2+z) = 0$. To satisfy the DC gain criterion we need $K_d z = 11$. Therefore $K_d = 5$ and $z = 2.2$. So the compensator is $K(s) = 5(s+2.2)$.

e) The root-locus is shown below.



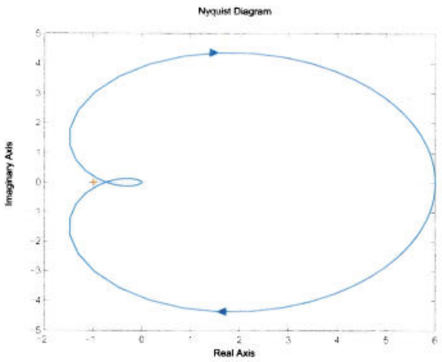
f) The error signal is given by

$$e(s) = \frac{r(s)}{1+G(s)K(s)}$$

Using the final value theorem for $r(s) = 1/s$ gives

$$e_{ss} = \frac{1}{1+G(0)K(0)} = \frac{1}{12}.$$

4. a) The Nyquist plot is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ to zero. This gives intercepts at $\boxed{\omega_i = 0, \pm\sqrt{3}, \infty}$ and so $\boxed{G(j\omega_i) = 6, -0.75, -0.75, 0.}$



b) The number of unstable closed-loop poles is determined by the number of encirclements by $G(s)$ of the point -1, which is zero. Thus the closed-loop is stable since $G(s)$ has no unstable poles. Since the real-axis intercept is at -0.75, the gain margin is $4/3$. For the phase margin, we need the intercept with the unit circle centred on the origin. We solve $|G(j\omega)| = 1$, this gives $\omega_1 \sqrt{6^{\frac{2}{3}} - 1}$ and $\arg[G(j\omega_1)] \approx -190°$. The phase margin is then $\approx 10°$.

c) The phase-lead has gain close to 1 for $\omega < \omega_0$ and close to $\frac{\omega_p}{\omega_0} > 1$ for $\omega > \omega_p$. The phase is positive and large between ω_0 and ω_p but small elsewhere. Thus the gain increase for $\omega > \omega_p$ degrades stability margins while the phase-lead increases the phase margin. It is important to balance the destabilizing increase in gain and the stabilizing increase in phase. We should place w_p and w_0 in the crossover frequency range (when $|G(j\omega)| \approx 1$).

d) One way of getting an infinite gain margin is to to reduce the order of $G(s)$ from 3 to 2. This can be done using the PD compensator $K(s) = K_d(s+1)$ (which is a special type of phase-lead compensator) since the zeros cancels one of the poles of $G(s)$. Taking $K_d = 1$ (to preserve the DC gain of $G(s)$), a sketch of the Nyquist diagram is given below.